# Residues of Weyl Group Multiple Dirichlet Series

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# Contents

**Abstract**

We give explicit computations of a pair of double Dirichlet series first studied by the Friedberg, Hoffstein, and Lieman. These computations are performed in the rational function field because, in this context, the series are power series which turn out to be rational functions.

Recently the Weyl group multiple Dirichlet series have been an area of intense research. We provide an explicit computation for these series associated to the root system of type $A_r$, again in the rational function field case. This shows the equivalence of two different descriptions of the local parts of these series and establishes uniqueness. It is conjectured that a multiresidue of the Weyl group multiple Dirichlet series correspond with the double Dirichlet series first noted of Friedberg, Hoffstein, and Lieman. This conjecture is proven in the rational function field case using the computed rational functions.

# Chapter 1

# Introduction

*joy unfound*

*in lovely sunset;*

*I turn and see the symmetry*

## 1.1 Background

The techniques of multiple Dirichlet series have arisen in recent decades as a way of providing estimates and average values of special values of $L$-functions over various fields. In just the past 10 years, there has been a concerted effort to unify and extend these results in a consistent framework. This has resulted in the description of Weyl group multiple Dirichlet series by Brubaker, Bump, Chinta, Friedberg, Hoffstein, and Gunnells in the papers [BBC$^+$06, BBF06, CGa] (by some subsets of the aforementioned authors). These are a generalization of single variable Dirichlet series to multiple complex variables and they satisfy a larger group of functional equations.

There remain a number of unanswered questions in these and related papers. For example, Friedberg, Hoffstein, and Lieman described two bivariate multiple Dirichlet series in [FHL02] which, taken together, satisfy a group of $32$ functional equations. These bivariate series are not actually Weyl group multiple Dirichlet series, but Brubaker and Bump ([BB06]) provided evidence that they are actually $(r-2)$-fold residues of Weyl group multiple Dirichlet series associated to the root system $A_r$ for $r \geq 2$. They verified this conjecture for type $A_3$, but were unable to prove the conjectured relationship in general. The case $r = 3$ is manageable because, thanks to Patterson [Pat77a, Pat77b], we have a complete understanding of the Fourier coefficients of the theta function on the 3-fold metaplectic cover of $SL_2$ that arise when we take the residues of the Dirichlet series constructed from cubic Gauss sums. For $r > 3$ the precise nature of the coefficients of the $r$-fold cover theta functions remains mysterious. Nevertheless, there is much evidence in favor of the expectation that the two series constructed by Friedberg, Hoffstein and Lieman coincide with a multiresidue of a Weyl group multiple Dirichlet series. The principal work of this dissertation is to explicitly compute these series over a rational function field and to show that these residue relationships hold for $r \geq 2$. In [Chi08] Chinta has carried out this explicit computation for $r = 2$. This dissertation expands on his work by computing these rational functions for all $A_r$ with $r \geq 2$.

Another partially open question is to understand the relationship of the construction in [BBC$^+$06, BBF06] and an alternative construction by Chinta and Gunnells in [CGa]. Chinta and Gunnells construct Weyl group multiple Dirichlet series by defining a group action and finding an invariant expression by summing over the Weyl group. The equivalence of these definitions has been supported by computational comparison and is proven rigorously for series of type $A_2$ built from quadratic twists in [CFG08]. In this dissertation we extend this result to show that these definitions are equivalent for stable series of type $A_r$.

In general, a multiple Dirichlet series has the form

$$Z(s_1, \ldots, s_r) = \sum \frac{a(c_1, \ldots, c_r)}{|c_1|^{s_1}|c_2|^{s_2} \cdots |c_r|^{s_r}},$$

where the sum is over the integer $r$-tuples $(c_1, \ldots, c_r)$ (or integer ring of some global field modulo units), $|\cdot|$ is the absolute norm. For the Weyl group multiple Dirichlet series, the numerator $a(c_1, \ldots, c_r)$ satisfies a twisted multiplicativity, initially we have convergence for $\Re(s_i)$ sufficiently large, and the analytic continuation has been established for all of $\mathbb{C}^r$. More specifically, the series corresponding to the root system $A_2$ is heuristically given by

$$Z_2^{(n)}(s_1, s_2) = \sum \frac{g(1, c_1) g(1, c_2) \left(\frac{c_1}{c_2}\right)^{-1}}{|c_1|^{s_1} |c_2|^{s_2}}.$$

We say "heuristically" since $\left(\frac{c_1}{c_2}\right)$ is only defined for $c_1, c_2$ relatively prime. In reality we define the numerator precisely by defining it for prime powers $c_1 = p^k$ and $c_2 = p^l$ and using the twisted multiplicativity. Here $g(1, c)$ is a Gauss sum constructed from $n^{\text{th}}$ power residue symbols and $\left(\frac{c_1}{c_2}\right)$ is such an $n^{\text{th}}$ power residue symbol. Each root system $A_r$ has an associated series with $r$ complex variables, which can be constructed from $n^{\text{th}}$ order Gauss sums for $n \geq 1$. We will indicate such a series by $Z_r^{(n)}$. We will give a more precise definition for $Z_r^{(n)}$ in Section 1.3.

The advantage of limiting ourselves to the rational function field is that the series $Z_r^{(n)}$ can be computed explicitly as a rational function of $|c_i|^{-s_i}$ for $1 \leq i \leq r$. With this explicit form, we can then use elementary techniques to evaluate the $r - 2$ residues and observe the desired relationship.

## 1.2 Common Definitions

Throughout this dissertation, $n \geq 2$ will be an integer and we construct our various series with $n^{\text{th}}$ power residue symbols. We will work over the finite field $\mathbb{F}_q$ with $q$ elements. Let $\mu_n = \{a \in \mathbb{F}_q : a^n = 1\}$ and let $\chi : \mathbb{F}_q^\times \to \mu_n$ be the character $a \mapsto a^{\frac{q-1}{n}}$. Let $K$ be the rational function field $\mathbb{F}_q(t)$ with polynomial ring $\mathcal{O}_K = \mathbb{F}_q[t]$. We let $K_\infty$ denote the field of Laurent series in $t^{-1}$.

In order to define Gauss sums we first need an additive character on $K_\infty$. Let $e_0$ be a nontrivial additive character on $\mathbb{F}_p$. Use $e_0$ to define a character $e_\star$ of $\mathbb{F}_q$ by $e_\star(a) =$

$e_0(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} a)$. Let $\omega$ be the global differential $dx/x^2$. Finally define the character $e$ of $K_\infty$ by $e(y) = e_\star(\mathrm{Res}_\infty(\omega y))$ for $y \in K_\infty$. Note that

$$\{y \in K : e|y\mathcal{O}_K = 1\} = \mathcal{O}_K.$$

Fix an embedding $\epsilon$ from the the $n^{th}$ roots of unity of $\mathbb{F}_q$ to $\mathbb{C}^\times$.

The most basic Gauss sums we utilize are

$$g(1,\epsilon,\mathfrak{p}) = \sum_{y \bmod \mathfrak{p}} \epsilon\left(\left(\frac{y}{p}\right)\right) e\left(\frac{y}{p}\right) \tag{1.1}$$

and

$$\tau(\epsilon) = \sum_{j \in \mathbb{F}_q} \epsilon\left(j^{(q-1)/n}\right) e_0\left(j\right), \tag{1.2}$$

which is associated with the field $\mathbb{F}_q$.

A common feature appearing throughout this dissertation is a formal relationship between the rational function defining the local part and the closed form evaluation of the series. This will appear in various guises, but the general variable changes

$$\begin{array}{rcll} s_i & \rightarrow & 2 - s_i & \text{for } 1 \le i \le r \\ |\mathfrak{p}| & \rightarrow & \frac{1}{q} & \\ g(1,\epsilon^i,\mathfrak{p})/\sqrt{|\mathfrak{p}|} & \rightarrow & \tau(\epsilon^i)/\sqrt{q} & \text{for } 1 \le i \le r \end{array} \tag{1.3}$$

transform the local rational function at $\mathfrak{p}$ to the global series $Z$. This similarity arises as the local series and the global series both satisfy functional equations with a similar form. We will show that these similar functional equations admit unique solutions, which must also be similar.

**Remark 1.2.1.** *The classical single variable $\zeta$ function associated with $\mathbb{F}_q[t]$ exhibits the variable transformations of equation (1.3). Recall that*

$$\begin{aligned} \zeta_{\mathbb{F}_q[t]}(s) &= \sum_{d \text{ monic}} |d|^{-s} = \sum_{n=0}^{\infty} \frac{\{\#\text{ monic polynomials of degree } n\}}{q^{ns}} \\ &= \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1-q^{1-s}}, \end{aligned} \tag{1.4}$$

*and the* $\mathfrak{p}$*-part for some irreducible* $\mathfrak{p}$ *is*

$$\zeta_{\mathfrak{p}}(s) = \sum_{n=0}^{\infty} |\mathfrak{p}^n|^{-s} = \frac{1}{1-|\mathfrak{p}|^{-s}}. \tag{1.5}$$

## 1.3 Defining $Z_r^{(n)}$ and Principal Results

In this section we will give a more precise definition of the Weyl group multiple Dirichlet series $Z_r^{(n)}$. We will also indicate the theorems which motivate the work in this dissertation. We define

$$Z_r^{(n)}(s_1,\ldots,s_r) = \Omega(s_1,\ldots,s_r) \sum \frac{H(c_1,\ldots,c_r)}{|c_1|^{s_1}|c_2|^{s_2}\cdots|c_r|^{s_r}} \tag{1.6}$$

where the sum is over all $r$-tuples $(c_1,\ldots,c_r)$ with $c_i$, $1 \le i \le r$, monic polynomials in $\mathbb{F}_q[t]$ and $\Omega$ is a product of normalizing zeta factors which is entirely defined in Chapter 4. The numerator $H(c_1,\ldots,c_r)$ is defined via a *twisted multiplicativity* which we now describe. For fixed $(c_1\cdots c_r, c_1'\cdots c_r') = 1$, we put

$$H(c_1c_1',\ldots,c_rc_r') = \xi(\mathbf{c},\mathbf{c}')H(c_1,\ldots,c_r)H(c_1',\ldots,c_r'), \tag{1.7}$$

where

$$\xi(\mathbf{c},\mathbf{c}') = \prod_{i=1}^{r}\left(\frac{c_i}{c_i'}\right)\left(\frac{c_i'}{c_i}\right)\prod_{i=2}^{r}\left(\frac{c_i}{c_{i-1}'}\right)\left(\frac{c_i'}{c_{i-1}}\right). \tag{1.8}$$

The $\mathfrak{p}$-part of $Z_r^{(n)}$ is given in Theorem 3.1.1. This $\mathfrak{p}$-part together with the twisted multiplicativity entirely determines the coefficients $H$.

Here is a brief outline of the remainder of this dissertation. In Chapter 2 we compute a pair of double Dirichlet series over the rational function field. We heuristically define the first

$$Z_{1,FHL}(s,w) = \sum_m \frac{L(s,\chi_m)}{|m|^w}$$

but delay any discussion of the second $Z_{2,FHL}$ until later. Here $\chi_m$ is an $n^{\text{th}}$ order power residue symbol. The series $Z_{1,FHL}$ and $Z_{2,FHL}$ are related by a group of functional equations induced by the functional equation of $L(s,\chi_m)$.

Chapter 3 lays the foundation for Chapter 4 by establishing properties of the $\mathfrak{p}$-part of $Z_r^{(n)}$. Following [CGa] we use a functional equation which the $\mathfrak{p}$-part $H_r^{(n)}(s_1,\ldots,s_r;\mathfrak{p})$ must satisfy to derive an explicit rational function expression for $H_r^{(n)}$. This establishes the following theorem:

**Theorem 1.3.1.** *Let $n > r$. Then the $\mathfrak{p}$-part of the Weyl group multiple Dirichlet series $Z_r^{(n)}$ described in [BBC+06] matches that of [CGa].*

Refer to Theorem 3.1.1 for the rational function.

Finally, in Chapter 4 we show that the $Z_{i,FHL}$ series are a multiresidue of Weyl group multiple Dirichlet series. We prove the following theorem:

**Theorem 1.3.2.** *Given $Z_r^{(r)}$ defined in equation (4.1) and $Z_{1,FHL}, Z_{2,FHL}$ defined in Chapter 2, we have*

$$\operatorname*{Res}_{x_2 \to q^{-(r+1)/r}} \cdots \operatorname*{Res}_{x_{r-1} \to q^{-(r+1)/r}} Z_r^{(r)}(x_1,x_2,\ldots,x_r) = \mathcal{E}_r \frac{Z_{1,FHL}(q^{1/r}x_1, q^{1/r}x_r)}{\prod_{i=2}^{r-1}(1-q^{r-i+2}x_1^r)(1-q^{r-i+2}x_r^r)} \tag{1.9}$$

*and*

$$\operatorname*{Res}_{x_3 \to q^{-(r+1)/r}} \cdots \operatorname*{Res}_{x_r \to q^{-(r+1)/r}} Z_r^{(r)}(x_1,x_2,\ldots,x_r) = \mathcal{E}_r \frac{Z_{2,FHL}(q^{1/2}x_1, q^{(r+1)/r}x_2)}{\prod_{i=2}^{r-1}(1-q^{r-i+1}x_1^r x_2^r)(1-q^{r-i+2}x_2^r)} \tag{1.10}$$

*where $\mathcal{E}_r$ is a constant depending only on $r$. We have used the notation $x_i = q^{-s_i}$, $1 \le i \le r$.*

Theorem 4.1.1 specifies the constant $\mathcal{E}_r$.

# Chapter 2

# A Pair of Double Dirichlet Series

*to comprehend*

*the equal equals;*

*we seek thy bashful face*

## 2.1 Introduction

The main result of this chapter is the explicit computation of an infinite sum of $L$-functions associated to $n^{th}$ order Hecke characters of $K$. The infinite sums we consider are examples of double Dirichlet series in two complex variables, and can be written as power series in $q^{-s}$ and $q^{-w}$. In fact it will turn out that the series we construct will be rational functions in $q^{-s}$ and $q^{-w}$.

The series studied in this chapter are not Weyl group multiple Dirichlet series. However, we will show in Chapter 4 that they are a multiresidue of certain Weyl group multiple Dirichlet series. We compute them here in the rational function field case in preparation for these later results.

These series are function field analogs of the series studied by Friedberg, Hoffstein and Lieman in [FHL02]. In that paper, working over a number field $F$ containing the $n^{th}$ roots of unity, the authors study a double Dirichlet series that is roughly of the form

$$\sum_m L(s, \chi_m)(Nm)^{-w},$$

where the sum is over integral ideals $m$ of $F$, the character $\chi_m$ is the $n^{th}$ order power residue symbol associated to $m$, and $Nm$ denotes the absolute norm. The authors show that this double Dirichlet series has a meromorphic continuation to all $(s, w) \in \mathbb{C}^2$ and satisfies a group of functional equations relating it to a second series constructed from Gauss sums. The main ingredients in the proof are the functional equation of $L(s, \chi_m)$, properties of the Fourier coefficients of the metaplectic Eisenstein series on the $n$-fold cover of $GL_2$, and Bochner's tube theorem.

In the case $n = 2$, these ideas were applied by Fisher and Friedberg [FF04] in the context of a general function field to show the rationality of double Dirichlet series constructed from quadratic $L$-functions. The case $n = 2$ is somewhat easier because the Gauss sum arising in the functional equation of a quadratic Hecke $L$-series is trivial, and the theory of metaplectic Eisenstein series is not needed.

Here we follow a more elementary method originally introduced in [CFH06] in the case $n = 2$. We exploit the fact that

$$\sum_{\substack{d \in \mathbb{F}_q[t] \\ \deg d = k}} \left(\frac{d}{m}\right)$$

vanishes if $k$ is bigger than or equal to the degree of $m$, unless $m$ is a perfect $n^{th}$ power. Here $\left(\frac{d}{m}\right) = \chi_m(d)$ denotes the $n^{th}$ power residue symbol for $m, d$ relatively prime. If $m$ and $d$ are monic, then we have the reciprocity law

$$\left(\frac{m}{d}\right) = \left(\frac{d}{m}\right), \tag{2.1}$$

when $q$ is congruent to $1$ mod $2n$, see e.g. Rosen [Ros02], Theorem 3.5.

We now describe our results more precisely. We will define two double Dirichlet series, explicitly compute them as rational functions in $q^{-s}, q^{-w}$ and show that they satisfy functional equations that relate them to one another. We begin by defining two multiplicative weighting factors $a(d,m)$ and $b(d,m)$ for pairs of monic polynomials, as in [FHL02]. For a monic prime polynomial $\mathfrak{p}$, let

$$a(\mathfrak{p}^j,\mathfrak{p}^k)=\begin{cases}|\mathfrak{p}|^{(n-1)d/n} & \text{if } d=\min(j,k) \text{ and } d\equiv 0 \bmod n\\ 0 & \text{otherwise,}\end{cases} \tag{2.2}$$

and

$$b(\mathfrak{p}^j,\mathfrak{p}^k)=\begin{cases}1 & \text{if } k=0\\ |\mathfrak{p}|^{k/2-1}(|\mathfrak{p}|-1) & \text{if } j\geq k, k\equiv 0 \bmod n, k>0\\ -|\mathfrak{p}|^{k/2-1} & \text{if } j=k-1, k\equiv 0 \bmod n, k>0\\ |\mathfrak{p}|^{(k-1)/2} & \text{if } j=k-1, k\not\equiv 0 \bmod n, k>0\\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

Then define

$$a(d,m)=\prod_{\substack{\mathfrak{p}^j||d\\ \mathfrak{p}^k||m}} a(\mathfrak{p}^j,\mathfrak{p}^k),\quad b(d,m)=\prod_{\substack{\mathfrak{p}^j||d\\ \mathfrak{p}^k||m}} b(\mathfrak{p}^j,\mathfrak{p}^k).$$

Here $|d|$ denotes the norm $q^{\deg d}$.

Let $\mathcal{O}$ denote $\mathbb{F}_q[t]$ and $\mathcal{O}_{mon}$ the set of monic polynomials in $\mathbb{F}_q[t]$. Let $\zeta_{\mathcal{O}}(s)$ be the zeta function of the ring $\mathcal{O}$, that is

$$\zeta_{\mathcal{O}}(s)=(1-q^{1-s})^{-1}.$$

The first double Dirichlet series we consider is

$$Z_1(s,w)=\sum_{d,m\in\mathcal{O}_{mon}}\frac{\chi_{m_0}(\hat{d})a(d,m)}{|m|^w|d|^s}, \tag{2.4}$$

where $m_0$ is the $n^{th}$ powerfree part of $m$ and $\hat{d}$ is the part of $d$ relatively prime to $m_0$. We show in Section 2.2 that this can be rewritten in terms of $L$-functions

$$Z_1(s,w)=\sum_{m\in\mathcal{O}_{mon}}\frac{L(s,\chi_{m_0})}{|m|^w}P(s;m), \tag{2.5}$$

where the $P(s;m)$ are finite Euler products defined in Proposition 2.2.1.

The second multiple Dirichlet series is built from Gauss sums. See Section 2.2 for the precise definition of the Gauss sum $g(r, \epsilon, \chi_m)$. Then

$$Z_2(s,w) = \zeta_{\mathcal{O}}(nw - \frac{n}{2} + 1) \sum_{d,m \in \mathcal{O}_{mon}} \frac{g(1,\epsilon,\chi_{m_0})}{\sqrt{|m_\flat|}} \frac{\bar{\chi}_{m_0}(\hat{d}) b(d,m)}{|m|^w |d|^s} \tag{2.6}$$

where $m_\flat$ is the squarefree part of the $n^{\mathrm{th}}$ powerfree part of $m$.

We can now state the main theorems of this chapter. The first describes a set of functional equations relating $Z_1$ and $Z_2$. Specifically, define

$$Z_1(s,w;\delta_i) = \sum_{\substack{d,m \in \mathcal{O}_{mon} \\ \deg m \equiv i \pmod n}} \frac{\chi_{m_0}(\hat{d}) a(d,m)}{|m|^w |d|^s}$$

and

$$Z_2(s,w;\delta_i) = \zeta_{\mathcal{O}}(nw - \frac{n}{2} + 1) \sum_{\substack{d,m \in \mathcal{O}_{mon} \\ \deg m \equiv i \pmod n}} \frac{g(1,\epsilon,\chi_{m_0})}{\sqrt{|m_\flat|}} \frac{\bar{\chi}_{m_0}(\hat{d}) b(d,m)}{|m|^w |d|^s}.$$

**Theorem 2.1.1.** *We have the functional equation*

$$Z_1(s,w;\delta_i) = \begin{cases} q^{2s-1} \frac{1-q^{-s}}{1-q^{s-1}} Z_2(1-s, w+s-\frac{1}{2}; \delta_0) & \text{for } i = 0 \\ q^{2s-1} q^{1/2-s} \frac{\bar{\tau}(\epsilon^i)}{\sqrt{q}} Z_2(1-s, w+s-\frac{1}{2}; \delta_i) & \text{for } 0 < i < n. \end{cases}$$

*The finite field Gauss sum $\tau(\epsilon^i)$ is defined in equation (1.2).*

This is proved in Section 2.4.

The second main theorem is the following:

**Theorem 2.1.2.** *The double Dirichlet series $Z_1$ and $Z_2$ are rational functions of $x = q^{-s}$ and $y = q^{-w}$. Explicitly,*

$$Z_1(s,w) = \frac{1 - q^2 xy}{(1-qx)(1-qy)(1-q^{n+1}x^n y^n)}, \tag{2.7}$$

*and*

$$Z_2(s,w) = \frac{1 - q^{3n/2} x^{n-1} y^n + \sum_{i=1}^{n-1} \left( \tau(\epsilon^i) q^{i-1+i/2} x^{i-1} y^i - \tau(\epsilon^i) q^{3i/2} x^i y^i \right)}{(1-qx)(1-q^{n/2+1}y^n)(1-q^{3n/2}x^n y^n)}. \tag{2.8}$$

This theorem is proved in Section 2.6.

When we have need outside of this chapter, we will refer to the series $Z_1$ and $Z_2$ as $Z_{1,FHL}$ and $Z_{2,FHL}$ respectively. Likewise, their local parts will be denoted as $H_{1,FHL}$ and $H_{2,FHL}$. Outside of this chapter, we reserve the notation $Z_r$ for the Weyl group multiple Dirichlet series associated with the root system $A_r$.

Finally we point out a curious connection between the series $Z_i$ of Theorem 2.1.2 and their $\mathfrak{p}$-parts. Define the following generating series $H_i$ constructed from the respective $\mathfrak{p}$-parts of $Z_1$ and $Z_2$:

$$\begin{aligned} H_1(X,Y) &= \sum_{j,k\geq 0} a(\mathfrak{p}^j,\mathfrak{p}^k)X^jY^k, \text{ and} \\ H_2(X,Y) &= (1-|\mathfrak{p}|^{n/2-1}Y^n)^{-1}\sum_{j,k\geq 0} b(\mathfrak{p}^j,\mathfrak{p}^k)\frac{g(1,\epsilon,\chi_{\mathfrak{p}^k})}{\sqrt{|\mathfrak{p}_\flat^k|}}X^jY^k, \end{aligned} \tag{2.9}$$

where $X=|\mathfrak{p}|^{-s}$ and $Y=|\mathfrak{p}|^{-w}$. We will prove

$$\begin{aligned} H_1(X,Y) &= \frac{1-XY}{(1-X)(1-Y)(1-|\mathfrak{p}|^{n-1}X^nY^n)}, \\ H_2(X,Y) &= \frac{1-|\mathfrak{p}|^{n/2-1}X^{(n-1)}Y^n+\sum_{i=1}^{n-1}\frac{g(1,\epsilon^i,\chi_\mathfrak{p})}{\sqrt{|\mathfrak{p}|}}X^{(i-1)}Y^i|\mathfrak{p}|^{(i-1)/2}(1-X)}{(1-X)(1-|\mathfrak{p}|^{n/2-1}Y^n)(1-|\mathfrak{p}|^{n/2}X^nY^n)}. \end{aligned} \tag{2.10}$$

Note that the substitutions

$$\begin{aligned} X &\to qx, \\ Y &\to qy, \\ |\mathfrak{p}| &\to 1/q, \text{ and} \\ g(1,\epsilon^i,\mathfrak{p})/\sqrt{|\mathfrak{p}|} &\to \tau(\epsilon^i)/\sqrt{q} \quad \text{for } 1\leq i\leq r \end{aligned} \tag{2.11}$$

transform $H_i$ into $Z_i$ for $i=1,2$. This is one manifestation of the general variable transformations given in equation (1.3).

## 2.2 Gauss sums and $L$-Functions

In this section we will define the Gauss sums and $L$-functions that are the constituents of our double Dirichlet series. We will mostly follow the notation of Patterson [Pat 2] but with some adjustments to facilitate comparison with [FHL02].

We now define a more general Gauss sum than the one found in the introduction in equation (1.1). For any $c \in \mathcal{O}$, we will use $c_0$ to indicate the $n^{\text{th}}$-power free part of $c$ and $c_\flat$ for the squarefree part of $c_0$. For $r, c \in \mathcal{O}$ we define the Gauss sum

$$g(r, \epsilon, \chi_c) = \sum_{y \bmod c_\flat} \epsilon\left(\left(\frac{y}{c}\right)\right) e\left(\frac{ry}{c_\flat}\right).$$

We also need the Gauss sums associated to the finite field $\mathbb{F}_q$. These are defined by

$$\tau(\epsilon) = \sum_{j \in \mathbb{F}_q} \epsilon\left(j^{(q-1)/n}\right) e_0\left(j\right).$$

We define the $L$-function associated to $\chi_m$ by

$$L(s, \chi_m) = \sum_{d \in \mathcal{O}_{mon}} \chi_m(d)|d|^{-s}. \tag{2.12}$$

When $m$ is $n^{\text{th}}$-power free, the $L$-function satisfies a functional equation that we will describe now. Denote the conductor of the character $\chi_m$ by cond $\chi_m$. Thus

$$|\text{cond } \chi_m| = \begin{cases} |m_\flat| & \text{deg } m \equiv 0(n) \\ q|m_\flat| & \text{deg } m \not\equiv 0(n). \end{cases}$$

Then the completed $L$-function

$$L^*(s, \chi_m) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_m) & \text{deg } m \equiv 0(n) \\ L(s, \chi_m) & \text{deg } m \not\equiv 0(n). \end{cases} \tag{2.13}$$

satisfies the functional equation

$$L^*(s, \chi_m) = q^{2s-1} |\text{cond } \chi_m|^{1/2-s} \frac{g^*(1, \epsilon, \chi_m)}{|\text{cond } \chi_m|^{1/2}} L^*(1-s, \bar{\chi}_m) \tag{2.14}$$

where

$$g^*(1,\epsilon,\chi_m) = \begin{cases} g(1,\epsilon,\chi_m) & \text{deg } m \equiv 0 \text{ (mod } n) \\ \bar{\tau}(\epsilon^i)g(1,\epsilon,\chi_m) & \text{deg } m \equiv i \not\equiv 0 \text{ (mod } n). \end{cases}$$

From the functional equation, we see that $L(s,\chi_m)$ is a polynomial in $q^{-s}$ whose degree is one less than the degree of $m_\flat$, if $m$ is not a perfect $n^{\text{th}}$ power. If $m = 1$, we recover the zeta function

$$\zeta_{\mathcal{O}}(s) = \sum_{d \in \mathcal{O}_{mon}} |d|^{-s} = \frac{1}{1-q^{1-s}}. \tag{2.15}$$

Expanding the components at infinity, we have the following functional equations when $m$ is $n^{\text{th}}$-power free, deg $m \equiv i$ (mod $n$):

$$L(s,\chi_m) = \begin{cases} q^{2s-1}|m_\flat|^{1/2-s} \frac{g(1,\epsilon,\chi_m)}{|m_\flat|^{1/2}} \frac{1-q^{-s}}{1-q^{-(1-s)}} L(1-s,\bar{\chi}_m) & i = 0 \\ q^{2s-1}(q|m_\flat|)^{1/2-s} \frac{\bar{\tau}(\epsilon^i)}{\sqrt{q}} \frac{g(1,\epsilon,\chi_m)}{|m_\flat|^{1/2}} L(1-s,\bar{\chi}_m) & 0 < i < n. \end{cases} \tag{2.16}$$

This functional equation will be used in Section 2.4 to relate $Z_1$ and $Z_2$.

We now introduce a modified $L$-function related to (2.12) by inserting the weighting factor $a(d,m)$. Define

$$L(s,\hat{\chi}_m) = \sum_{d \in \mathcal{O}_{mon}} \frac{\chi_{m_0}(\hat{d})a(d,m)}{|d|^s}, \tag{2.17}$$

where $\hat{d}$ is the part of $d$ relatively prime to $m_0$. Since the weighting function is multiplicative, $L(s,\hat{\chi}_m)$ is an Euler product,

$$L(s,\hat{\chi}_m) = \prod_{\substack{\mathfrak{p} \in \mathcal{O}_{mon} \\ \text{irreducible}}} \left(1 + \frac{\chi_{m_0}(\hat{\mathfrak{p}})a(\mathfrak{p},m)}{|\mathfrak{p}|^s} + \frac{\chi_{m_0}(\hat{\mathfrak{p}}^2)a(\mathfrak{p}^2,m)}{|\mathfrak{p}|^{2s}} + \ldots\right).$$

Further, since $a(d,m) = 1$ when $d$ and $m$ are coprime, this Euler product agrees with the original $L$-function Euler product for all but finitely many places.

We will relate this modified $L$-function $L(s,\hat{\chi}_m)$ to $L(s,\chi_{m_0})$ and derive a bound on its degree as a polynomial in $q^{-s}$, as long as $m$ is not a perfect $n^{\text{th}}$-power. These properties are given in the following proposition:

**Proposition 2.2.1.** *We have*

$$L(s,\hat{\chi}_m) = L(s,\chi_{m_0})P(s;m),$$

*where* $P(s;m) = \prod_{\mathfrak{p}} P_{\mathfrak{p}}(s;m)$ *and* $P_{\mathfrak{p}}(s;m) =$

$$\begin{cases} \left(1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}\right)\displaystyle\sum_{k=0}^{n\alpha-1}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}} + |\mathfrak{p}|^{-n\alpha s}|\mathfrak{p}|^{(n-1)\alpha} & \text{if } \mathfrak{p}\nmid m_0 \\ \displaystyle\sum_{k=0}^{n\alpha}\frac{a(\mathfrak{p}^{n\alpha+i},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}} & \text{if } \mathfrak{p}^i||m_0 \text{ and } i\neq 0. \end{cases}$$

*Here* $\alpha$ *and* $i$ *are the unique integers with* $0\le i<n$ *and* $\mathfrak{p}^{n\alpha+i}\|m$*. In particular, for* $m$ *not a perfect* $n^{\text{th}}$ *power, the degree of* $L(s,\hat{\chi}_m)$ *as a polynomial in* $q^{-s}$ *is less than the degree of* $m$*.*

*Proof.* Begin with the Euler product

$$\begin{aligned} L(s,\hat{\chi}_m) &= \prod_{\mathfrak{p}}\sum_{k=0}^{\infty}\frac{\chi_{m_0}(\hat{\mathfrak{p}}^k)a(m,\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}} \\ &= \prod_{\mathfrak{p}^{n\alpha}||m}\sum_{k=0}^{\infty}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}} \times \prod_{\substack{\mathfrak{p}^{n\alpha+i}||m \\ 0<i<n}}\sum_{k=0}^{\infty}\frac{a(\mathfrak{p}^{n\alpha+i},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}} \end{aligned}$$

For primes $\mathfrak{p}$ with $i=0$—that is $\mathfrak{p}\nmid m_0$ and $\mathfrak{p}^{n\alpha}||m$, say— it follows from (2.2) that the tails of the sum are a geometric series with common ratio $\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}$. Thus for such $\mathfrak{p}$ the $\mathfrak{p}$-part is

$$\sum_{k=0}^{n\alpha-1}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}} + \frac{|\mathfrak{p}|^{-n\alpha s}|\mathfrak{p}|^{(n-1)\alpha}}{1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}} = \left(1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}\right)^{-1}P_{\mathfrak{p}}(s;m),$$

where

$$P_{\mathfrak{p}}(s;m) = \sum_{k=0}^{n\alpha-1}\frac{\chi_{m_0}(\mathfrak{p}^k)a(\mathfrak{p}^{n\alpha},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}\left(1-\chi_{m_0}(\mathfrak{p})|\mathfrak{p}|^{-s}\right) + |\mathfrak{p}|^{-n\alpha s}|\mathfrak{p}|^{(n-1)\alpha}. \quad (2.18)$$

For primes such that $\mathfrak{p}^i||m_0$ with $0<i<n$, it follows from (2.2) that $a(\mathfrak{p}^{n\alpha+i},\mathfrak{p}^k) = 0$ for $k>n\alpha$, so the $\mathfrak{p}$-part is a finite sum

$$P_{\mathfrak{p}}(s;m) = \sum_{k=0}^{n\alpha}\frac{a(\mathfrak{p}^{n\alpha+i},\mathfrak{p}^k)}{|\mathfrak{p}|^{ks}}.$$

Thus

$$L(s, \hat{\chi}_m) = L(s, \chi_{m_0})P(s; m)$$

as claimed. The bound on the degree of $L(s, \hat{\chi}_m)$ follows from the bound on the degree of $L(s, \chi_{m_0})$ for $m_0 \neq 1$ and the degrees of the $P_p(s; m)$. □

## 2.3 Functional Equation: $H_1 \to H_2$

Recall that the generating series $H_1(X, Y)$ and $H_2(X, Y)$ of (2.9) define the $p$-parts of $Z_1$ and $Z_2$, respectively. We describe the functional equations relating $H_1(X, Y)$ and $H_2(X, Y)$. These will be used to prove the global functional equation relating $Z_1$ to $Z_2$.

The functional equations are a direct consequence of the following proposition:

**Proposition 2.3.1.** *The generating series $H_1(X, Y)$ and $H_2(X, Y)$ are rational functions of $X$ and $Y$. Explicitly,*

$$H_1(X, Y) = \frac{1 - XY}{(1 - X)(1 - Y)(1 - |\mathfrak{p}|^{n-1}X^nY^n)}, \tag{2.19}$$

*and*

$$H_2(X, Y) = \frac{1 - |\mathfrak{p}|^{n/2-1}X^{(n-1)}Y^n + \sum_{i=1}^{n-1} \frac{g(1, \epsilon^i, \chi_{\mathfrak{p}})}{\sqrt{|\mathfrak{p}|}} X^{(i-1)}Y^i|\mathfrak{p}|^{(i-1)/2}(1 - X)}{(1 - X)(1 - |\mathfrak{p}|^{n/2-1}Y^n)(1 - |\mathfrak{p}|^{n/2}X^nY^n)}. \tag{2.20}$$

*Proof.* Equation (2.19) is obvious from the definition (2.2) of the $a(\mathfrak{p}^k, \mathfrak{p}^l)$. The evaluation of $H_2(X, Y)$ is simply a matter of recognizing geometric series. From the definitions of $b(\mathfrak{p}^j, \mathfrak{p}^k)$ in (2.3) and $H_2$ in (2.9), we have

$$\begin{aligned}(1-|\mathfrak{p}|^{n/2-1}Y^n)H_2(X,Y) =& \sum_{j=0}^{\infty} \frac{g(1,\epsilon,\chi_{\mathfrak{p}^0})}{\sqrt{|\mathfrak{p}_\flat^0|}} X^j Y^0 \\ &+ \sum_{\alpha=1}^{\infty}\sum_{j=n\alpha}^{\infty} |\mathfrak{p}|^{n\alpha/2-1}(|\mathfrak{p}|-1)\frac{g(1,\epsilon,\chi_{\mathfrak{p}^{n\alpha}})}{\sqrt{|\mathfrak{p}_\flat^{n\alpha}|}} X^j Y^{n\alpha} \\ &+ \sum_{\alpha=1}^{\infty} -|\mathfrak{p}|^{n\alpha/2-1}\frac{g(1,\epsilon,\chi_{\mathfrak{p}^{n\alpha}})}{\sqrt{|\mathfrak{p}_\flat^{n\alpha}|}} X^{n\alpha-1} Y^{n\alpha} \\ &+ \sum_{\alpha=0}^{\infty}\sum_{i=1}^{n-1} |\mathfrak{p}|^{(n\alpha+i-1)/2}\frac{g(1,\epsilon,\chi_{\mathfrak{p}^{n\alpha+i}})}{\sqrt{|\mathfrak{p}_\flat^{n\alpha+i}|}} X^{n\alpha+i-1} Y^{n\alpha+i}.\end{aligned}$$

Evaluating the geometric series yields

$$\begin{aligned}(1-|\mathfrak{p}|^{n/2-1}Y^n)H_2(X,Y) =& \frac{1}{1-X} + \frac{|\mathfrak{p}|^{n/2-1}(|\mathfrak{p}|-1)X^nY^n}{(1-X)(1-|\mathfrak{p}|^{n/2}X^nY^n)} \\ &+ \frac{-|\mathfrak{p}|^{n/2-1}X^{n-1}Y^n}{1-|\mathfrak{p}|^{n/2}X^nY^n} + \sum_{i=1}^{n-1} \frac{\frac{g(1,\epsilon^i,\chi_\mathfrak{p})}{\sqrt{|\mathfrak{p}|}}|\mathfrak{p}|^{(i-1)/2}X^{i-1}Y^i}{(1-|\mathfrak{p}|^{n/2}X^nY^n)}.\end{aligned} \tag{2.21}$$

Equation (2.20) follows by rewriting equation (2.21). □

For $0 \le i < n$, define

$$\begin{aligned} H_1(X,Y;\delta_i) &= \sum_{\substack{j,k\ge 0\\ k\equiv i(n)}} a(\mathfrak{p}^j,\mathfrak{p}^k)X^jY^k, \\ H_2(X,Y;\delta_i) &= (1-|\mathfrak{p}|^{n/2-1}Y^n)^{-1}\sum_{\substack{j,k\ge 0\\ k\equiv i(n)}} b(\mathfrak{p}^j,\mathfrak{p}^k)\frac{g(1,\epsilon,\chi_{\mathfrak{p}^k})}{\sqrt{|\mathfrak{p}_\flat^k|}}\bar{\chi}_{\mathfrak{p}^k}(\hat{\mathfrak{p}}^j)X^jY^k.\end{aligned} \tag{2.22}$$

We have shown in Proposition 2.3.1 that $H_1$ and $H_2$ are both rational functions in $|\mathfrak{p}|^{-s}$ and $|\mathfrak{p}|^{-w}$ and it is clear from this proposition that

$$H_1(X,Y;\delta_i) = \begin{cases} \frac{1-XY^n}{(1-X)(1-Y^n)(1-|\mathfrak{p}|^{n-1}X^nY^n)} & i\equiv 0 \\ \frac{(1-X)Y^i}{(1-X)(1-Y^n)(1-|\mathfrak{p}|^{n-1}X^nY^n)} & i\not\equiv 0, \end{cases} \tag{2.23}$$

and

$$H_2(X,Y;\delta_i) = \begin{cases} \frac{1-|\mathfrak{p}|^{n/2-1}X^{(n-1)}Y^n}{(1-X)(1-|\mathfrak{p}|^{n/2-1}Y^n)(1-|\mathfrak{p}|^{n/2}X^nY^n)} & i\equiv 0 \\ \frac{\frac{g(1,\epsilon^i,\chi_\mathfrak{p})}{\sqrt{|\mathfrak{p}|}}X^{(i-1)}Y^i|\mathfrak{p}|^{(i-1)/2}(1-X)}{(1-X)(1-|\mathfrak{p}|^{n/2-1}Y^n)(1-|\mathfrak{p}|^{n/2}X^nY^n)} & i\not\equiv 0. \end{cases} \tag{2.24}$$

The following theorem establishes a functional equation relating $H_1$ to $H_2$:

**Theorem 2.3.2.** *We have the functional equation*

$$H_1(\mathfrak{p}^{-s}, \mathfrak{p}^{-w}; \delta_i) = \begin{cases} \frac{1-|\mathfrak{p}|^{-(1-s)}}{1-|\mathfrak{p}|^{-s}} H_2(\mathfrak{p}^{-(1-s)}, \mathfrak{p}^{-(w+s-1/2)}; \delta_0) & i = 0 \\ \frac{\sqrt{|\mathfrak{p}|}}{g(1,\epsilon^i,\chi_\mathfrak{p})} |\mathfrak{p}|^{s-1/2} H_2(\mathfrak{p}^{-(1-s)}, \mathfrak{p}^{-(w+s-1/2)}; \delta_i) & 0 < i < n. \end{cases}$$

*Proof.* The proof is by a direct computation using Equations (2.23) and (2.24). □

## 2.4 Functional Equation: $Z_1 \to Z_2$

There is a set of functional equations relating $Z_1$ and $Z_2$. These will be described in this section. Define

$$Q(s; m) = \frac{P(1-s; m)}{(m/m_\flat)^{s-1/2}}. \tag{2.25}$$

With the expansion of $P$ as an Euler product, we see that $Q$ is also an Euler product supported in the primes dividing $m$:

$$Q(s; m) = \prod_{\substack{\mathfrak{p}^{n\alpha+i}||m \\ i=0}} \frac{1}{|\mathfrak{p}|^{n\alpha(s-1/2)}} P_\mathfrak{p}(1-s; m) \times \prod_{\substack{\mathfrak{p}^{n\alpha+i}||m \\ 0<i<n}} \frac{1}{|\mathfrak{p}|^{(n\alpha+i-1)(s-1/2)}} P_\mathfrak{p}(1-s; m).$$

**Proposition 2.4.1.** *Define*

$$Z_2'(s, w) = \sum_m \frac{g(1, \epsilon, \chi_{m_0})}{\sqrt{|m_\flat|}} \frac{L(s, \bar{\chi}_{m_0}) Q(s; m)}{|m|^w}.$$

*We have* $Z_2' = Z_2$.

*Proof.* Define

$$H_2'(\mathfrak{p}^{-s}, \mathfrak{p}^{-w}; \delta_i) = \begin{cases} (1-\mathfrak{p}^{-s})^{-1} \sum_{k=0}^\infty \frac{Q(s;\mathfrak{p}^{nk})}{\mathfrak{p}^{nkw}} & i = 0 \\ \frac{g(1,\epsilon^i,\chi_\mathfrak{p})}{\sqrt{|\mathfrak{p}|}} \sum_{k=0}^\infty \frac{Q(s;\mathfrak{p}^{nk+i})}{\mathfrak{p}^{nkw}} & 0 < i < n. \end{cases} \tag{2.26}$$

Then $H_2'(\mathfrak{p}^{-s}, \mathfrak{p}^{-w}) = \sum_{i=0}^{n-1} H_2'(\mathfrak{p}^{-s}, \mathfrak{p}^{-w}; \delta_i)$ is the $\mathfrak{p}$-part of $Z_2'$. We will show that $H_2'$ and $H_2$ both satisfy the functional equations with $H_1$ shown in Theorem 2.3.2 and therefore $H_2' = H_2$. The result follows since, for fixed $m$, the $L$-functions, $P$, and $Q$ each have Euler products.

As a result of the definition in equation (2.25), the $\mathfrak{p}$-parts of $Q$ and $P$ satisfy

$$P(s;\mathfrak{p}^{n\alpha+i}) = \begin{cases} \mathfrak{p}^{n\alpha(1/2-s)} Q(1-s;\mathfrak{p}^{n\alpha}) & i=0 \\ \mathfrak{p}^{(n\alpha+i-1)(1/2-s)} Q(1-s;\mathfrak{p}^{n\alpha+i}) & 0<i<n. \end{cases}$$

Therefore, we relate $H_1$ to $H_2'$ by

$$\begin{aligned} H_1(\mathfrak{p}^{-s},\mathfrak{p}^{-w};\delta_0) =& (1-|\mathfrak{p}|^{-s})^{-1} \sum_{k=0}^{\infty} \frac{P(s;\mathfrak{p}^{nk})}{|\mathfrak{p}|^{nkw}} \\ =& (1-|\mathfrak{p}|^{-s})^{-1} \sum_{k=0}^{\infty} \frac{Q(1-s;\mathfrak{p}^{nk})|\mathfrak{p}|^{nk(1/2-s)}}{|\mathfrak{p}|^{nkw}} \\ =& (1-|\mathfrak{p}|^{-s})^{-1} \sum_{k=0}^{\infty} \frac{Q(1-s;\mathfrak{p}^{nk})}{|\mathfrak{p}|^{nk(w+s-1/2)}} \\ =& \frac{1-|\mathfrak{p}|^{-(1-s)}}{1-|\mathfrak{p}|^{-s}} H_2'(\mathfrak{p}^{-(1-s)},\mathfrak{p}^{-(w+s-1/2)};\delta_0). \end{aligned}$$

This is exactly the functional equation satisfied by $H_2$ in Theorem 2.3.2. A similar computation shows that

$$H_1(\mathfrak{p}^{-s},\mathfrak{p}^{-w};\delta_i) = \frac{\sqrt{|\mathfrak{p}|}}{g(1,\epsilon^i,\chi_{\mathfrak{p}})} |\mathfrak{p}|^{s-1/2} H_2(\mathfrak{p}^{-(1-s)},\mathfrak{p}^{-(w+s-1/2)};\delta_i)$$

for $0<i<n$. Thus $H_2'(X,Y;\delta_i) = H_2(X,Y;\delta_i)$ for all $0 \le i < n$ and this completes the proof. $\square$

For $0 \le i < n$, define

$$Z_1(s,w;\delta_i) = \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m \equiv i \pmod{n}}} \frac{L(s,\chi_{m_0}) P(s;m)}{|m|^w}$$

and

$$Z_2(s,w;\delta_i) = \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m \equiv i \pmod{n}}} \frac{g(1,\epsilon,\chi_{m_0})}{\sqrt{|m_\flat|}} \frac{L(s,\bar{\chi}_{m_0}) Q(s;m)}{|m|^w}.$$

**Theorem 2.4.2.** *We have the functional equation*

$$Z_1(s,w;\delta_i) = \begin{cases} q^{2s-1}\frac{1-q^{-s}}{1-q^{s-1}}Z_2(1-s,w+s-\frac{1}{2};\delta_0) & \text{for } i=0 \\ q^{2s-1}q^{1/2-s}\frac{\bar{\tau}(\epsilon^i)}{\sqrt{q}}Z_2(1-s,w+s-\frac{1}{2};\delta_i) & \text{for } 0<i<n. \end{cases}$$

*Proof.* This is a direct computation utilizing the functional equation (2.16) for $L(s,\chi_{m_0})$.

□

## 2.5 Convolutions

We define a convolution operation $\star$ on rational functions in $x$ and $y$ with power series expansions around the origin. For

$$A(x,y) = \sum_{j,k\geq 0} a(j,k)x^j y^k \text{ and } B(x,y) = \sum_{j,k\geq 0} b(j,k)x^j y^k,$$

define

$$(A\star B)(x,y) = \sum_{j,k\geq 0} a(j,k)b(j,k)x^j y^k.$$

We can compute convolutions as the double integral

$$(A\star B)(x,y) = \left(\frac{1}{2\pi i}\right)^2 \int\int A(u,v)B(\frac{x}{u},\frac{y}{v})\frac{dudv}{uv}, \tag{2.27}$$

where each integral is a counterclockwise circuit of a small circle in the complex plane. (The circle must be small enough that $A(x,y)$ is holomorphic for $x,y$ inside the circle.) We will utilize the residue theorem to compute this contour integral.

## 2.6 Evaluation of $Z_1$ and $Z_2$

We will now prove Theorem 2.1.2. We first establish the identity (2.7); then (2.8) will follow from the functional equation (2.1.1). It follows from Proposition 2.2.1 that

$$\sum_{\substack{d\in\mathcal{O}_{mon} \\ \deg d=k}} \chi_{m_0}(\hat{d})a(d,m) = 0 \tag{2.28}$$

when deg $m \leq k$ unless $m$ is a perfect $n^{\text{th}}$ power. To prove (2.7) of Theorem 2.1.2, we begin by writing

$$Z_1(s,w) = Z_a(s,w) + Z_a(w,s) - Z_b(s,w) \tag{2.29}$$

where

$$Z_a(s,w) = \sum_{k \geq j \geq 0} \frac{1}{q^{jw} q^{ks}} \sum_{\substack{d,m \in \mathcal{O}_{mon} \\ \deg m = j \\ \deg d = k}} \chi_{m_0}(\hat{d}) a(d,m)$$

and

$$Z_b(s,w) = \sum_{k \geq 0} \frac{1}{q^{kw} q^{ks}} \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m = j}} \sum_{\substack{d \in \mathcal{O}_{mon} \\ \deg d = k}} \chi_{m_0}(\hat{d}) a(d,m).$$

First, note that

$$\sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m = j}} \sum_{\substack{d \in \mathcal{O}_{mon} \\ \deg d = k}} \chi_{m_0}(\hat{d}) a(d,m) = \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m = j}} \sum_{\substack{d \in \mathcal{O}_{mon} \\ \deg d = k}} \chi_{d_0}(\hat{m}) a(m,d).$$

When $m$ and $d$ are coprime, the reciprocity law (2.1) guarantees that $\chi_{m_0}(\hat{d}) = \chi_{d_0}(\hat{m})$. Otherwise, when $m$ and $d$ are not coprime, $\chi_{m_0}(\hat{d}) \neq \chi_{d_0}(\hat{m})$ only when there exists a prime $\mathfrak{p}$ such that $\mathfrak{p} | d_0$ and $\mathfrak{p} | m_0$. In this case $a(d,m) = 0$. The symmetry $a(d,m) = a(m,d)$ is obvious. This establishes the validity of the decomposition (2.29) of $Z_1$.

Now the key observation is that because of equation (2.28), we have

$$Z_a(s,w) = \sum_{k \geq j \geq 0} \frac{1}{q^{jw} q^{ks}} \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m = j \\ m_0 = 1}} \sum_{\substack{d \in \mathcal{O}_{mon} \\ \deg d = k}} a(d,m)$$

and

$$Z_b(s,w) = \sum_{k \geq 0} \frac{1}{q^{kw} q^{ks}} \sum_{\substack{m \in \mathcal{O}_{mon} \\ \deg m = k \\ m_0 = 1}} \sum_{\substack{d \in \mathcal{O}_{mon} \\ \deg d = k}} a(d,m),$$

that is, the inner sum vanishes unless $m$ is a perfect $n^{\text{th}}$-power. This leads us to consider the series

$$T_a(s,w) = \sum_{\substack{m \in \mathcal{O}_{mon} \\ m_0 = 1}} \sum_{d \in \mathcal{O}_{mon}} \frac{a(d,m)}{|m|^w |d|^s},$$

which has an Euler product

$$\begin{aligned}
T_a(s,w) &= \prod_{\mathfrak{p}} \sum_{j,k\geq 0} \frac{a(\mathfrak{p}^j,\mathfrak{p}^{nk})}{|\mathfrak{p}|^{nk}|\mathfrak{p}|^j} \\
&= \prod_{\mathfrak{p}} \frac{1-|\mathfrak{p}|^{-s-nw}}{(1-|\mathfrak{p}|^{-s})(1-|\mathfrak{p}|^{-nw})(1-|\mathfrak{p}|^{(n-1)-ns-nw})} \\
&= \frac{\zeta(s)\zeta(nw)\zeta(ns+nw-(n-1))}{\zeta(s+nw)} \\
&= \frac{1-q^{1-s-nw}}{(1-q^{1-s})(1-q^{1-nw})(1-q^{n-ns-nw})} \\
&= \frac{1-qxy^n}{(1-qx)(1-qy^n)(1-q^nx^ny^n)} \\
&= \tilde{T}_a(x,y),
\end{aligned} \tag{2.30}$$

with $x=q^{-s}, y=q^{-w}$.

It is clear that

$$Z_a(s,w) = (\tilde{T}_a \star \tilde{K}_a)(x,y) \tag{2.31}$$

where

$$\begin{aligned}
\tilde{K}_a(x,y) &= \frac{1}{(1-x)(1-xy)} \\
&= \sum_{j\geq k\geq 0} x^j y^k.
\end{aligned}$$

Compute the convolution in (2.31) by means of the integral in equation (2.27) which we can evaluate using the residue theorem. We find

$$Z_a(s,w) = \frac{1}{(1-q^{n+1}x^ny^n)(1-qx)}. \tag{2.32}$$

We can compute $Z_b$ similarly: let $\tilde{K}_b(x,y) = \frac{1}{1-xy}$. Then

$$\begin{aligned}
Z_b(s,w) &= (\tilde{T}_a \star \tilde{K}_b)(x,y) \\
&= \frac{1}{1-q^{n+1}x^ny^n}.
\end{aligned} \tag{2.33}$$

Putting this all together,

$$
\begin{aligned}
Z_1(s,w) &= \frac{1}{(1-q^{n+1}x^ny^n)(1-qx)} + \frac{1}{(1-q^{n+1}x^ny^n)(1-qy)} \\
&\quad - \frac{1}{1-q^{n+1}x^ny^n} \\
&= \frac{1-qy+1-qx-(1-qy)(1-qx)}{(1-q^{n+1}x^ny^n)(1-qx)(1-qy)} \\
&= \frac{1-q^2xy}{(1-q^{n+1}x^ny^n)(1-qx)(1-qy)}.
\end{aligned}
\tag{2.34}
$$

This establishes (2.7).

With the rational function for $Z_1(s,w)$, we can use the functional equations relating $Z_1(s,w;\delta_i)$ and $Z_2(s,w;\delta_i)$ for $0 \le i < n$ to evaluate $Z_2(s,w)$. Expanding the geometric series $\frac{1}{1-qy}$ and collecting terms with the exponent on $y$ congruent to $i$ (mod $n$), we arrive at

$$
Z_1(s,w;\delta_i) = \begin{cases} \frac{1-q^{n+1}xy^n}{(1-qx)(1-q^ny^n)(1-q^{n+1}x^ny^n)} & i=0 \\ \frac{(q^i-q^{i+1}x)y^i}{(1-qx)(1-q^ny^n)(1-q^{n+1}x^ny^n)} & 0<i<n. \end{cases}
$$

Using the functional equations relating $Z_1(s,w,\delta_i)$ and $Z_2(1-s,w+s-\frac{1}{2},\delta_i)$ and remembering that $\left|\frac{\tau(\epsilon^i)}{\sqrt{q}}\right| = 1$, we see that

$$
Z_2(s,w;\delta_i) = \begin{cases} q^{2s-1}\frac{1-q^{-s}}{1-q^{s-1}} Z_1(1-s,w+s-\frac{1}{2};\delta_i) & i=0 \\ q^{2s-1}q^{1/2-s}\frac{\tau(\epsilon^i)}{\sqrt{q}} Z_1(1-s,w+s-\frac{1}{2};\delta_i) & 0<i<n. \end{cases}
$$

With this in hand, $Z_2$ is

$$
q^{2s-1}\frac{1-q^{-s}}{1-q^{s-1}} Z_1(1-s,w+s-\frac{1}{2};\delta_0) + \sum_{i=1}^{n-1} q^{2s-1}q^{1/2-s}\frac{\tau(\epsilon^i)}{\sqrt{q}} Z_1(1-s,w+s-\frac{1}{2};\delta_i).
\tag{2.35}
$$

When simplified, equation (2.35) is the rational function for $Z_2$ given in Theorem 2.1.2.

# Chapter 3

# Uniqueness of Local Factors

*an axiom and acorn;*

*both humble brothers*

*give tender shade for weary bones*

## 3.1 Introduction

Prior to attacking our ultimate goal of proving a statement about an $(r-2)$-fold residue of the Weyl group multiple Dirichlet series associated to the the root system of type $A_r$ built from $r^{\text{th}}$ order Gauss sums, we must describe the local part of this series associated to the prime $\mathfrak{p}$. The global series is then built from these local pieces by a twisted multiplicativity described in Chapter 1. The goal of the current chapter is to derive this $\mathfrak{p}$-part from a set of functional equations. Philosophically, this follows the methods of Chinta and Gunnells in [CG07, CGa] and, with Friedberg, [CFG08]. This method stands in contrast to the earlier combinatorial description given by Brubaker, Bump, Chinta, Friedberg, and Hoffstein in [BBC$^+$06, BBF06]. We briefly describe these two approaches here focusing on the so-called stable case, which is our main interest in this dissertation. For root systems of type $A_r$, the series falls into the stable case precisely when it is constructed from $n^{\text{th}}$ order Gauss sums when $n \geq r$. Our principal interest is when $n = r$.

The original description given by Brubaker et al. gives the stable coefficients of the numerator of $\mathfrak{p}$-part in the following way. Fix a decomposition of the root system $\Phi = A_r$ into positive and negative roots, $\Phi = \Phi^+ \cup \Phi^-$, let $\alpha_1, \ldots, \alpha_r$ be the simple roots and let $W$ be the Weyl group generated by reflections taking $\alpha_i$ to $-\alpha_i$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. A point $(k_1, \ldots, k_r) \in \mathbb{Z}^r$ is called *stable* if $w \in W$ and $\rho - w\rho = \sum_{i=1}^{r} k_i\alpha_i$. The stable points form the vertices of a convex polytope called the *permutahedron* attached to $\Phi$. In the stable case, the coefficients of the numerator have the following properties:

1. the coefficient associated to stable vertex $(k_1, \ldots, k_r)$ is a product of $l(w)$ Gauss sums of order $n$ where $l(w)$ is the length of $w$,
2. all other coefficients are $0$, and
3. therefore, there are $|W|$ non-zero coefficients.

The method of Chinta and Gunnells starts from an axiomatic description of the $\mathfrak{p}$-part by giving a functional equation that the $\mathfrak{p}$-part must satisfy. This functional equation originates from work of Hoffstein in [Hof 1] and is applied to the $\mathfrak{p}$-parts by performing the variable changes of equation (1.3) in reverse. Chinta and Gunnells construct the $\mathfrak{p}$-part by performing a sum over the Weyl group

$$H(\mathbf{x}) = \sum_{w\in W} \frac{1}{\Delta(w\mathbf{x})}(1|w)(\mathbf{x}) \tag{3.1}$$

where $\Delta(w\mathbf{x})$ is a normalizing factor and $(1|w)$ is the rational function action described in equation (3.16), which incorporates the functional equation.

In theory, all of the coefficients can be found by solving a linear system that we derive in this chapter. However, this linear system has so many variables that it is unlikely to be efficient computationally without further refinement. The advantage is that the development here is elementary, although the functional equation may appear unmotivated initially. Both of these methods of constructing the global series utilize the twisted multiplicativity described in equation (1.7).

In [BBF06] and [Hof 1], the poles of the series $Z_r^{(n)}$ are established in a very general context. The denominator $D(\mathbf{x})$ which we define now reflects this understanding:

$$D(\mathbf{x}) = \prod_{\alpha\in\Phi^+} (1 - p^{n|\alpha|-1}\mathbf{x}^{n\alpha}). \tag{3.2}$$

In order to state the next theorem, we introduce a notation which we use in Theorem 3.1.1 and in Chapter 4. Let

$$\Theta = \left\{ \frac{r}{2}, \frac{r}{2} - 1, \ldots, -\frac{r}{2} + 1, -\frac{r}{2} \right\},$$

then we can think of elements of the symmetric group $w \in S_{r+1}$ as bijective functions $w : \Theta \to \Theta$. We write $w_i = w(i)$.

Our technique proceeds as follows. Instead of summing over the Weyl group as in [CGa], we derive the coefficients explicitly in the stable case for root systems of type $A_r$. This results in an explicit description that matches that of [BBC$^+$06]. We also see that this $\mathfrak{p}$-part is unique up to a constant. Thus, the rational function we derive here is forced to match the $\mathfrak{p}$-part found by Chinta and Gunnells, and we have the following theorem:

**Theorem 3.1.1.** *Let $W$ be the Weyl group of the root system $A_r$, that is $W$ is the symmetric group $S_{r+1}$. The $\mathfrak{p}$-part of the Weyl group multiple Dirichlet series $Z_r^{(n)}$ described in [BBC$^+$06] matches that of [CGa]. Letting $\mathbf{x}$ be the vector $(x_1, \ldots, x_r) = (|\mathfrak{p}|^{-s_1} \ldots, |\mathfrak{p}|^{-s_r})$, this $\mathfrak{p}$-part has the following form*

$$H_r^{(n)}(\mathbf{x}) = \frac{\sum_{w\in W} G(w;\mathfrak{p})\mathbf{x}^{\rho - w\rho}}{D(\mathbf{x})} \tag{3.3}$$

*where*

$$G(w;\mathfrak{p}) = \prod_{\substack{i<j \\ w_i<w_j}} g(\mathfrak{p}^{w_j-w_i-1}, \epsilon, \mathfrak{p}^{w_j-w_i}). \tag{3.4}$$

We will prove this in Section 3.4 as a result of Theorems 3.4.4 and 3.4.5.

In the rational function field case, the global Weyl group multiple Dirichlet series turns out to be a rational function. Since the functional equation described in this chapter is related to the functional equation of the global series given in [BBC$^+$06] by the variable changes of equation (1.3), we can deduce that global Weyl group multiple Dirichlet series matches the form of the $\mathfrak{p}$-part.

## 3.2 $A_r$ Weyl Group

In general, Weyl group multiple Dirichlet series have a group of functional equations isomorphic to the Weyl group of a root system. A classical example of this is the Riemann zeta function, which has a group of functional equations of order 2. This group is isomorphic to the Weyl group of the $A_1$ root system. This section collects the necessary background definitions and some minor technical lemmas about root systems. For the most part, we specialize the definitions and results of this section to the root system $A_r$. For a more general development we refer to [CGa], where the notation is similar.

In general, a root system $\Phi$ of rank $r$ has a basis of $r$ simple roots $\{\alpha_i\}_{i=1}^r$ which are vectors in Euclidean space. The Weyl group $W$ of $\Phi$ is the group generated by the simple reflections $\{\sigma_i\}_{i=1}^r$, where $\sigma_i$ reflects $\alpha_i$ across the hyperplane perpendicular to $\alpha$ sending $\alpha_i$ to $-\alpha_i$. The root system $\Phi$ is composed of the image of the simple roots under the action of $W$.

If $\Phi = A_r$, the $r$ simple roots have the following arrangement in Euclidean space. The angle between $\alpha_i$ and $\alpha_j$ is $\frac{2\pi}{3}$ if $|i-j| = 1$; otherwise $\alpha_i$ and $\alpha_j$ are orthogonal if $|i-j| > 1$. We say that $\alpha_i$ and $\alpha_j$ are *adjacent* if $|i-j| = 1$ and write $i \sim j$. The standard realization of $\Phi$ in $\mathbb{R}^{r+1}$ is given by setting

$$\alpha_i = e_i - e_{i+1}, \tag{3.5}$$

where $e_i$ is the $i^{\text{th}}$ standard basis vector. The reflection $\sigma_i$ acts on $\mathbb{R}^{r+1}$ by swapping the $i$ and $(i+1)^{\text{st}}$ entry. This is precisely the permutation representation of the symmetric group $S_{r+1}$ on $\mathbb{R}^{r+1}$ so that the Weyl group for $A_r$ is isomorphic to $S_{r+1}$. This will be very important here and in Chapter 4.

It is customary to write $\Phi$ as the disjoint union of positive and negative roots $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+$ contains the $r$ simple roots and $\Phi^+$ and $\Phi^-$ are separated by a hyperplane. The element $\rho$ appears frequently in this work, and is one-half the sum of the positive roots. Explicitly, for $\Phi = A_r$, we have

$$\rho = (\frac{r}{2}, \frac{r}{2} - 1, \ldots, -\frac{r}{2} + 1, -\frac{r}{2}).$$

As an alternative to linear combinations of the simple roots $\alpha_i$ defined in equation (3.5), we often work in terms of the root lattice $\Lambda$; the free abelian group generated by $\Phi$ in $\mathbb{R}^{r+1}$. An element $\beta \in \Lambda$ is this a sum $\sum_{i=1}^r \beta_i \alpha_i$, where $\beta_i \in \mathbb{Z}$. We define $d(\beta) = \sum_{i=1}^r \beta_i$ and $d_i(\beta) = \sum_{i \sim j} \beta_j$. In addition, we say that $\beta \succ 0$ if $\beta_i > 0$ for all $1 \leq i \leq r$, and for $\beta, \gamma \in \Lambda$, $\beta \succ \gamma$ if $\beta - \gamma \succ 0$.

In the rest of this section we record several actions of $W$ on the root lattice $\Lambda$. The generators $\sigma_i \in W$ satisfy

$$(\sigma_i \sigma_j)^{r(i,j)} = e, \text{ where } r(i,j) = \begin{cases} 1 & i = j \\ 3 & i \sim j \\ 2 & \text{otherwise.} \end{cases} \tag{3.6}$$

The standard reflection action of the Weyl group acts by

$$\sigma_i \alpha_j = \begin{cases} -\alpha_j & i = j \\ \alpha_i + \alpha_j & i \sim j \\ \alpha_j & \text{otherwise.} \end{cases} \tag{3.7}$$

Equation (3.7) implies that for $\beta \in \Lambda$ we have $\sigma_i \beta = \beta'$, where

$$\beta'_j = \begin{cases} \beta_j & j \neq i \\ d_i(\beta) - \beta_i & j = i. \end{cases} \tag{3.8}$$

In the context of multiple Dirichlet series it is natural to define a shifted reflection defined by

$$\sigma_i \cdot \beta = \rho - \sigma_i(\rho - \beta). \tag{3.9}$$

By explicit computation, one can verify that $w \cdot v \cdot \gamma = (wv) \cdot \gamma$ for all $w, v \in W$ and $\gamma \in \Lambda$. This shows that this action of the simple reflections extends to an action of the Weyl group $W$. On the root lattice $\Lambda$, this action takes the form $\sigma_i \cdot \beta = \beta'$ where

$$\beta'_j = \begin{cases} \beta_j & j \neq i \\ d_i(\beta) - \beta_i + 1 & j = i. \end{cases} \tag{3.10}$$

Observe that $\rho - w \cdot \rho = w\rho$ showing that the convex hull of the orbit $W \cdot 0$ is identical in size and shape to the orbit $W\rho$. We utilize this to show the following lemma, which will be useful in the following sections:

**Lemma 3.2.1.** *The intersection of any line parallel to a simple root and passing through any vertex of $W \cdot 0$ with the solid convex hull of $W \cdot 0$ has length at most $r$ times the Euclidean length of any simple root.*

*Proof.* As we have stated the image of $\rho$ under the natural reflection action of $W$ is exactly the image $W \cdot 0$ translated by $\rho$. The reflection action of $W$ acts on

$$\rho = \left\langle \frac{r}{2}, \frac{r-1}{2}, \ldots, \frac{-r-1}{2}, \frac{-r}{2} \right\rangle$$

by permutations and is generated by transpositions. We observe that any two elements of $\rho$ have a difference no greater than $\frac{r}{2} - \frac{-r}{2} = r$. For any element $w \in W$, the length $\sigma_i w\rho - w\rho \leq r|\alpha_i|$. The vertices $\sigma_i w\rho$ and $w\rho$ form the end-points of the intersection with the solid convex hull of $W\rho$ of a line parallel to a simple root. □

Let $m_i(\beta)$ denote the coefficient of $\alpha_i$ in $\sigma_i \cdot \beta - \beta$. One can compute that $m_i(\beta) = d_i(\beta) - 2\beta_i + 1$. Define $\mu_i(\beta) \in \mathbb{Z}$ and $\ell_i(\beta) \in \mathbb{Z}$ such that

$$m_i(\beta) = \ell_i(\beta)n + \mu_i(\beta), \quad 0 \leq \mu_i(\beta) < n.$$

**Proposition 3.2.2.** *The action of $\sigma_i$ on $\Lambda$ defined by*

$$\sigma_i \star \beta = \beta' \text{ where } \beta'_j = \begin{cases} \beta_j & j \neq i \\ \beta_i + \ell_i(\beta)n & j = i \end{cases} \tag{3.11}$$

*is an involution.*

*Proof.* It is clear that $\sigma_i \star (\sigma_i \star \beta)$ only differs from $\beta$ in the $i^{\text{th}}$ entry. The $i^{\text{th}}$ entry of $\sigma_i \star (\sigma_i \star \beta)$ is

$$\begin{aligned} \beta_i + \left\lfloor \frac{d_i(\beta) - 2\beta_i + 1}{n} \right\rfloor n + \left\lfloor \frac{d_i(\beta) - 2\left(\beta_i + \left\lfloor \frac{d_i(\beta)-2\beta_i+1}{n} \right\rfloor n\right) + 1}{n} \right\rfloor n \\ = \beta_i + \left\lfloor \frac{d_i(\beta) - 2\beta_i + 1}{n} \right\rfloor n + \left\lfloor \frac{d_i(\beta) - 2\beta_i + 1}{n} - 2\left\lfloor \frac{d_i(\beta) - 2\beta_i + 1}{n} \right\rfloor \right\rfloor n. \end{aligned}$$

Since $\lfloor x \rfloor + \lfloor x - 2\lfloor x \rfloor \rfloor = 0$ for all $x \in \mathbb{R}$, the final expression above equals $\beta_i$. □

**Remark 3.2.3.** *It can be shown that the action defined in equation (3.11) is an action of the Weyl group, but it is not necessary in this dissertation.*

## 3.3 An Action on Laurent Series

Let $A = C[\Lambda]$ be the ring of Laurent polynomials on the lattice $\Lambda$. Hence $A$ consists of all expressions of the form $f = \sum_{\beta \in \Lambda} c_\beta x^\beta$, where $c_\beta \in \mathbb{C}$ and almost all are zero, and the multiplication of monomials is defined by addition in $\Lambda$: $x^\beta x^\lambda = x^{\beta+\lambda}$. Given $f$, the set of $\{\beta | c_\beta \neq 0\}$ is called the support of $f$ and denoted by Supp $f$. We identify $A$ with $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ via $x^{\alpha_i} \mapsto x_i$.

Let $\mathbf{x} = (x_1, \ldots, x_r)$. First, we define an action on $\mathbf{x}$ by

$$\sigma_i \mathbf{x} = \mathbf{x}' \text{ where } \mathbf{x}'_j = \begin{cases} x_j & |j - i| > 1 \\ p x_j x_i & |j - i| = 1 \\ 1/p^2 x_i & j = i. \end{cases} \tag{3.12}$$

Again, this action of $\sigma_i$ extends to be an action of $W$ on vectors $\mathbf{x}$. It is clear that $\sigma_i\sigma_j\mathbf{x} = \sigma_j\sigma_i\mathbf{x}$ when $|i-j| > 2$. The rest of the requirements are seen by computing the following items:

1. Compute the $i, i+1, i+2$ components of $\sigma_i\sigma_j\mathbf{x}$ and $\sigma_j\sigma_i\mathbf{x}$ when $i+2=j$.

2. For $1 < i < r$, compute the $i-1, i, i+1$ components of $\sigma_i\sigma_i\mathbf{x}$. For $i=1$ and $i=r$, note that nothing fundamentally changes.

3. For $1 < i < r-1$, compute the $i-1, i, i+1, i+2$ components of $\sigma_i\sigma_{i+1}\sigma_i\mathbf{x}$ and compare to $\sigma_{i+1}\sigma_i\sigma_{i+1}\mathbf{x}$. For $i=1$ and $i=r-1$ perform the same comparison omitting the $i-1^{\text{st}}$ and $i+2^{\text{nd}}$ component respectively.

Now let $\Lambda' \subset \Lambda$ be the sublattice generated by the set $\{n\alpha\}_{\alpha\in\Phi}$. A direct computation with Cartan matrices shows that $W$ takes $\Lambda$ into itself. Let $\tilde{A}$ be the field of fractions of $A$. We have the decomposition

$$\tilde{A} = \bigoplus_{\lambda\in\Lambda/\Lambda} \tilde{A}_\lambda, \tag{3.13}$$

where $\tilde{A}_\lambda$ consists of the functions $f/g(f, g \in A)$ such that Supp $g$ lies in the kernel of the map $\nu : \Lambda \to \Lambda/\Lambda$, and $\nu$ maps Supp $f$ to $\lambda$.

Define the normalized Gauss sum

$$g^*(1, \epsilon^i, \mathfrak{p}) = \begin{cases} g(1, \epsilon^i, \mathfrak{p})/|\mathfrak{p}| & \text{if } i \not\equiv 0 \bmod n \\ -1 & \text{otherwise.} \end{cases} \tag{3.14}$$

Finally, we define a $W$-action on $A$. Let

$$\begin{aligned} P_{\beta,i}(x) &= (|\mathfrak{p}|x)^{1-\mu_i(\beta)} \frac{1-\frac{1}{|\mathfrak{p}|}}{1-|\mathfrak{p}|^{n-1}x^n} \text{ and} \\ Q_{\beta,i}(x) &= -g^*(1, \epsilon^{-\mu_i(\beta)}, \mathfrak{p})(|\mathfrak{p}|x)^{1-n} \frac{1-|\mathfrak{p}|^n x^n}{1-|\mathfrak{p}|^{n-1}x^n}. \end{aligned} \tag{3.15}$$

Define $A_\beta \subset A$ as the collection of rational functions in $A$ such that each monomial $c_\gamma \mathbf{x}^\gamma$ has $\gamma \equiv \beta \bmod n$. Then $W$ acts on $f \in A_\beta$ by

$$(f|\sigma_i)(\mathbf{x}) = P_{\beta,i}(x_i) f(\sigma_i \mathbf{x}) + Q_{\sigma_i \cdot \beta, i}(x_i) f(\sigma_i \mathbf{x}) \tag{3.16}$$

and the action extends to all of $A$ by linearity.

**Lemma 3.3.1.** *The action of $W = \langle \sigma_1, \ldots, \sigma_r \rangle$ on $A$ described in equation (3.16) satisfies the relations (3.6) so that equation (3.16) defines an action of $W$ on $A$.*

*Proof.* The main work of the proof is to establish certain rational function relations between the $P$ and $Q$ with various inputs when the action is applied to a monomial $f(\mathbf{x}) = \mathbf{x}^\beta$. This is carried out in detail with the help of tables that were constructed by computer. That this extends linearly to the entire space $A$ is made clear by the careful definition of the space $A$.

To complete these rational function verifications the only Gauss sum identity required is that

$$g^*(1, \epsilon^i, \mathfrak{p}) g^*(1, \epsilon^{-i}, \mathfrak{p}) = \begin{cases} 1 & n|i \\ \frac{1}{|\mathfrak{p}|} & n \nmid i. \end{cases}$$

In the tables in the figures below, we have simplified the notation by letting $g_i^* = g^*(1, \epsilon^i, \mathfrak{p})$ and $p = |\mathfrak{p}|$. We have also written $(i)_n$ to denote the remainder of $i$ when dividing by $n$ such $0 \leq (i)_n < n$.

Let $f \in F_\beta$, then we can see that $((f|\sigma_i)|\sigma_i) = f$ using the table in Figure 3.1. Each line of the table represents a product with two factors selected from $P$ and $Q$ that arise when applying $\sigma_i$ twice in succession. We want to show that the sum of the products of each row is exactly 1. In the case that $\mu_i(\beta) \neq 0$, we can see that

$$P_{\beta,i}(x_i) P_{\beta,i}(\sigma_i x_i) + Q_{\beta,i}(x_i) Q_{\sigma_i \cdot \beta, i}(\sigma_i x_i) = 1, \text{ and}$$

$$P_{\sigma_i \cdot \beta, i}(x_i) Q_{\sigma_i \cdot \beta, i}(\sigma_i x_i) + Q_{\sigma_i \cdot \beta, i}(x_i) P_{\beta,i}(\sigma_i x_i) = 0.$$

| | $\sigma_i$ | $\sigma_i$ |
|---|---|---|
| 1 | $P_{\beta,i}(x_i) = \frac{(px_i)^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ | $P_{\beta,i}(\sigma_i x_i) = \frac{(p(\frac{1}{p^2x_i}))^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(\frac{1}{p^2x_i})^n}$ |
| 2 | $P_{\sigma_i\cdot\beta,i}(x_i) = \frac{(px_i)^{1-(-m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_i x_i) = -g^*_{\mu_i(\beta)}\frac{(p(\frac{1}{p^2x_i}))^{1-n}\left(1-p^n(\frac{1}{p^2x_i})^n\right)}{1-p^{n-1}(\frac{1}{p^2x_i})^n}$ |
| 3 | $Q_{\sigma_i\cdot\beta,i}(x_i) = -g^*_{\mu_i(\beta)}\frac{(px_i)^{1-n}\left(1-p^nx_i^n\right)}{1-p^{n-1}x_i^n}$ | $P_{\beta,i}(\sigma_i x_i) = \frac{(p(\frac{1}{p^2x_i}))^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(\frac{1}{p^2x_i})^n}$ |
| 4 | $Q_{\sigma_i^2\cdot\beta,i}(x_i) = -g^*_{-\mu_i(\beta)}\frac{(px_i)^{1-n}\left(1-p^nx_i^n\right)}{1-p^{n-1}x_i^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_i x_i) = -g^*_{\mu_i(\beta)}\frac{(p(\frac{1}{p^2x_i}))^{1-n}\left(1-p^n(\frac{1}{p^2x_i})^n\right)}{1-p^{n-1}(\frac{1}{p^2x_i})^n}$ |

(3.17)

Figure 3.1: $P, Q$ computations for verifying involutions

Alternatively, if $\mu_i(\beta) = 0$ we can verify that the sum of the products represented in each row is $1$, but it is necessary to take all four rows together.

| | $\sigma_i$ | $\sigma_j$ | $\sigma_i$ |
|---|---|---|---|
| 1 | $P_{\beta,i}(x_i) = \frac{(px_i)^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ | $P_{\beta,j}(\sigma_i x_j) = \frac{(p(px_j x_i))^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_j x_i)^n}$ | $P_{\beta,i}(\sigma_i\sigma_j x_i) = \frac{(px_j)^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ |
| 2 | $P_{\sigma_i\cdot\beta,i}(x_i) = \frac{(px_i)^{1-(-m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ | $P_{\sigma_i\cdot\beta,j}(\sigma_i x_j) = \frac{(p(px_j x_i))^{1-(m_i(\beta)+m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_j x_i)^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_i\sigma_j x_i) = -g^*_{\mu_i(\beta)}\frac{(px_j)^{1-n}\left(1-p^n x_j^n\right)}{1-p^{n-1}x_j^n}$ |
| 3 | $P_{\sigma_j\cdot\beta,i}(x_i) = \frac{(px_i)^{1-(m_i(\beta)+m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ | $Q_{\sigma_j\cdot\beta,j}(\sigma_i x_j) = -g^*_{m_j(\beta)}\frac{(p(px_j x_i))^{1-n}(1-p^n(px_j x_i)^n)}{1-p^{n-1}(px_j x_i)^n}$ | $P_{\beta,i}(\sigma_i\sigma_j x_i) = \frac{(px_j)^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ |
| 4 | $P_{\sigma_j\cdot\sigma_i\cdot\beta,i}(x_i) = \frac{(px_i)^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ | $Q_{\sigma_j\cdot\sigma_i\cdot\beta,j}(\sigma_i x_j) = -g^*_{\mu_i(\beta)+m_j(\beta)}\frac{(p(px_j x_i))^{1-n}(1-p^n(px_j x_i)^n)}{1-p^{n-1}(px_j x_i)^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_i\sigma_j x_i) = -g^*_{\mu_i(\beta)}\frac{(px_j)^{1-n}\left(1-p^n x_j^n\right)}{1-p^{n-1}x_j^n}$ |
| 5 | $Q_{\sigma_i\cdot\beta,i}(x_i) = -g^*_{\mu_i(\beta)}\frac{(px_i)^{1-n}\left(1-p^n x_i^n\right)}{1-p^{n-1}x_i^n}$ | $P_{\beta,j}(\sigma_i x_j) = \frac{(p(px_j x_i))^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_j x_i)^n}$ | $P_{\beta,i}(\sigma_i\sigma_j x_i) = \frac{(px_j)^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ |
| 6 | $Q_{\sigma_i^2\cdot\beta,i}(x_i) = -g^*_{-\mu_i(\beta)}\frac{(px_i)^{1-n}\left(1-p^n x_i^n\right)}{1-p^{n-1}x_i^n}$ | $P_{\sigma_i\cdot\beta,j}(\sigma_i x_j) = \frac{(p(px_j x_i))^{1-(m_i(\beta)+m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_j x_i)^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_i\sigma_j x_i) = -g^*_{\mu_i(\beta)}\frac{(px_j)^{1-n}\left(1-p^n x_j^n\right)}{1-p^{n-1}x_j^n}$ |
| 7 | $Q_{\sigma_i\cdot\sigma_j\cdot\beta,i}(x_i) = -g^*_{\mu_i(\beta)+m_j(\beta)}\frac{(px_i)^{1-n}\left(1-p^n x_i^n\right)}{1-p^{n-1}x_i^n}$ | $Q_{\sigma_j\cdot\beta,j}(\sigma_i x_j) = -g^*_{m_j(\beta)}\frac{(p(px_j x_i))^{1-n}(1-p^n(px_j x_i)^n)}{1-p^{n-1}(px_j x_i)^n}$ | $P_{\beta,i}(\sigma_i\sigma_j x_i) = \frac{(px_j)^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ |
| 8 | $Q_{\sigma_i\cdot\sigma_j\cdot\sigma_i\cdot\beta,i}(x_i) = -g^*_{m_j(\beta)}\frac{(px_i)^{1-n}\left(1-p^n x_i^n\right)}{1-p^{n-1}x_i^n}$ | $Q_{\sigma_j\cdot\sigma_i\cdot\beta,j}(\sigma_i x_j) = -g^*_{\mu_i(\beta)+m_j(\beta)}\frac{(p(px_j x_i))^{1-n}(1-p^n(px_j x_i)^n)}{1-p^{n-1}(px_j x_i)^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_i\sigma_j x_i) = -g^*_{\mu_i(\beta)}\frac{(px_j)^{1-n}\left(1-p^n x_j^n\right)}{1-p^{n-1}x_j^n}$ |

(3.18)

Figure 3.2: $P, Q$ computations necessary to test the braid relation left hand side.

| | $\sigma_j$ | $\sigma_i$ | $\sigma_j$ |
|---|---|---|---|
| 1 | $P_{\beta,j}(x_j) = \frac{(px_j)^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ | $P_{\beta,i}(\sigma_j x_i) = \frac{(p(px_ix_j))^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_ix_j)^n}$ | $P_{\beta,j}(\sigma_j\sigma_i x_j) = \frac{(px_i)^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ |
| 2 | $P_{\sigma_j\cdot\beta,j}(x_j) = \frac{(px_j)^{1-(-m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ | $P_{\sigma_j\cdot\beta,i}(\sigma_j x_i) = \frac{(p(px_ix_j))^{1-(m_i(\beta)+m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_ix_j)^n}$ | $Q_{\sigma_j\cdot\beta,j}(\sigma_j\sigma_i x_j) = -g^*_{m_j(\beta)}\frac{(px_i)^{1-n}\left(1-p^nx_i^n\right)}{1-p^{n-1}x_i^n}$ |
| 3 | $P_{\sigma_i\cdot\beta,j}(x_j) = \frac{(px_j)^{1-(m_i(\beta)+m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_j x_i) = -g^*_{\mu_i(\beta)}\frac{(p(px_ix_j))^{1-n}(1-p^n(px_ix_j)^n)}{1-p^{n-1}(px_ix_j)^n}$ | $P_{\beta,j}(\sigma_j\sigma_i x_j) = \frac{(px_i)^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ |
| 4 | $P_{\sigma_i\cdot\sigma_j\cdot\beta,j}(x_j) = \frac{(px_j)^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_j^n}$ | $Q_{\sigma_i\cdot\sigma_j\cdot\beta,i}(\sigma_j x_i) = -g^*_{\mu_i(\beta)+m_j(\beta)}\frac{(p(px_ix_j))^{1-n}(1-p^n(px_ix_j)^n)}{1-p^{n-1}(px_ix_j)^n}$ | $Q_{\sigma_j\cdot\beta,j}(\sigma_j\sigma_i x_j) = -g^*_{m_j(\beta)}\frac{(px_i)^{1-n}\left(1-p^nx_i^n\right)}{1-p^{n-1}x_i^n}$ |
| 5 | $Q_{\sigma_j\cdot\beta,j}(x_j) = -g^*_{m_j(\beta)}\frac{(px_j)^{1-n}\left(1-p^nx_j^n\right)}{1-p^{n-1}x_j^n}$ | $P_{\beta,i}(\sigma_j x_i) = \frac{(p(px_ix_j))^{1-(m_i(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_ix_j)^n}$ | $P_{\beta,j}(\sigma_j\sigma_i x_j) = \frac{(px_i)^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ |
| 6 | $Q_{\sigma_j^2\cdot\beta,j}(x_j) = -g^*_{-m_j(\beta)}\frac{(px_j)^{1-n}\left(1-p^nx_j^n\right)}{1-p^{n-1}x_j^n}$ | $P_{\sigma_j\cdot\beta,i}(\sigma_j x_i) = \frac{(p(px_ix_j))^{1-(m_i(\beta)+m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}(px_ix_j)^n}$ | $Q_{\sigma_j\cdot\beta,j}(\sigma_j\sigma_i x_j) = -g^*_{m_j(\beta)}\frac{(px_i)^{1-n}\left(1-p^nx_i^n\right)}{1-p^{n-1}x_i^n}$ |
| 7 | $Q_{\sigma_j\cdot\sigma_i\cdot\beta,j}(x_j) = -g^*_{\mu_i(\beta)+m_j(\beta)}\frac{(px_j)^{1-n}\left(1-p^nx_j^n\right)}{1-p^{n-1}x_j^n}$ | $Q_{\sigma_i\cdot\beta,i}(\sigma_j x_i) = -g^*_{\mu_i(\beta)}\frac{(p(px_ix_j))^{1-n}(1-p^n(px_ix_j)^n)}{1-p^{n-1}(px_ix_j)^n}$ | $P_{\beta,j}(\sigma_j\sigma_i x_j) = \frac{(px_i)^{1-(m_j(\beta))n}\left(1-\frac{1}{p}\right)}{1-p^{n-1}x_i^n}$ |
| 8 | $Q_{\sigma_j\cdot\sigma_i\cdot\sigma_j\cdot\beta,j}(x_j) = -g^*_{\mu_i(\beta)}\frac{(px_j)^{1-n}\left(1-p^nx_j^n\right)}{1-p^{n-1}x_j^n}$ | $Q_{\sigma_i\cdot\sigma_j\cdot\beta,i}(\sigma_j x_i) = -g^*_{\mu_i(\beta)+m_j(\beta)}\frac{(p(px_ix_j))^{1-n}(1-p^n(px_ix_j)^n)}{1-p^{n-1}(px_ix_j)^n}$ | $Q_{\sigma_j\cdot\beta,j}(\sigma_j\sigma_i x_j) = -g^*_{m_j(\beta)}\frac{(px_i)^{1-n}\left(1-p^nx_i^n\right)}{1-p^{n-1}x_i^n}$ |

(3.19)

Figure 3.3: $P, Q$ computations necessary to test the braid relation right hand side.

To show that the braid relation is satisfied, we need to show that $f|\sigma_i\sigma_j\sigma_i = f|\sigma_j\sigma_i\sigma_j$ for $f \in F_\beta$. Figures 3.2 and 3.3 give the details about the rational functions involved in this verification. One can verify that the following conditioned identities hold. We suppress the subscripts on $P$ and $Q$, but note that the products on the left hand side of the equal sign come from Figure 3.2 and the products on the right hand side come from Figure 3.3. Independent of $\mu_i(\beta)$ and $\mu_j(\beta)$, we have

$$P(x_i)P(\sigma_i x_i)P(\sigma_i\sigma_j x_i) = P(x_i)P(\sigma_j x_i)P(\sigma_j\sigma_i x_i),$$

$$Q(x_i)Q(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) = Q(x_i)Q(\sigma_j x_i)Q(\sigma_j\sigma_i x_i),$$

$$P(x_i)Q(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) = Q(x_i)Q(\sigma_j x_i)P(\sigma_j\sigma_i x_i),$$

$$Q(x_i)Q(\sigma_i x_i)P(\sigma_i\sigma_j x_i) = P(x_i)Q(\sigma_j x_i)Q(\sigma_j\sigma_i x_i).$$

If $\mu_i(\beta) = \mu_j(\beta) = 0$, then

$$Q(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) = Q(x_i)P(\sigma_j x_i)Q(\sigma_j\sigma_i x_i),$$

$$P(x_i)Q(\sigma_i x_i)P(\sigma_i\sigma_j x_i) = P(x_i)Q(\sigma_j x_i)P(\sigma_j\sigma_i x_i),$$

$$P(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) + Q(x_i)P(\sigma_i x_i)P(\sigma_i\sigma_j x_i) =$$

$$P(x_i)P(\sigma_j x_i)Q(\sigma_j\sigma_i x_i) + Q(x_i)P(\sigma_j x_i)P(\sigma_j\sigma_i x_i).$$

Otherwise

$$P(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) + Q(x_i)P(\sigma_i x_i)P(\sigma_i\sigma_j x_i) + Q(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) =$$

$$P(x_i)Q(\sigma_i x_i)P(\sigma_i\sigma_j x_i) + Q(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i), \text{ and}$$

and

$$P(x_i)Q(\sigma_i x_i)P(\sigma_i\sigma_j x_i) + Q(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) =$$

$$P(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i) + Q(x_i)P(\sigma_i x_i)P(\sigma_i\sigma_j x_i) + Q(x_i)P(\sigma_i x_i)Q(\sigma_i\sigma_j x_i).$$

□

## 3.4 Invariant Rational Functions

We now examine the rational functions with the denominator $D(\mathbf{x})$ defined in equation (3.2) that are invariant under the action of equation (3.16). We will develop a linear system that the coefficients of the numerator $N(\mathbf{x})$ must satisfy if $f(\mathbf{x}) = \frac{N(\mathbf{x})}{D(\mathbf{x})}$ is to be invariant. Let

$$N(\mathbf{x}) = \sum_{\substack{\beta\in\Lambda \\ \beta\succ 0}} c_\beta \mathbf{x}^\beta \tag{3.20}$$

Substituting the expression for $f(\mathbf{x})$ into equation (3.16), we will derive a related functional equation that the numerator $N(\mathbf{x})$ must satisfy. Observe that the action of $\sigma_i$ defined in equation (3.12) permutes the set

$$\left\{p^{n|\alpha|-1}\mathbf{x}^{n\alpha} \mid \alpha \in \Phi^+\right\} \cup \left\{p^{-n|\alpha_i|-1}\mathbf{x}^{-n\alpha_i}\right\}.$$

This implies that the factors of $D(\mathbf{x})$ are permuted by $\sigma_i$ with the exception of $1 - p^{n|\alpha_i|-1}\mathbf{x}^{n\alpha_i}$. To accommodate this, we define

$$P^\diamond_{\beta,i}(x) = (px)^{1-\mu_i(\beta)} \frac{|\mathfrak{p}|^{n+1} x^n \left(1 - \frac{1}{p}\right)}{|\mathfrak{p}|^{n+1} x^n - 1}, \text{ and} \tag{3.21}$$

$$Q^\diamond_{\beta,i}(x) = -g^*(1, \epsilon^{-\mu_i(\beta)}, \mathfrak{p})(px)^{1-n} \frac{|\mathfrak{p}|^{n+1} x^n (1 - |\mathfrak{p}|^n x^n)}{|\mathfrak{p}|^{n+1} x^n - 1}. \tag{3.22}$$

With this definition, we have

$$(N|\sigma_i)(\mathbf{x}) = P^\diamond_{\beta,i}(x_i) N(\sigma_i \mathbf{x}) + Q^\diamond_{\sigma_i\cdot\beta,i}(x_i) N(\sigma_i \mathbf{x}) \tag{3.23}$$

Substituting $N(\mathbf{x})$ into equation (3.23)

$$\begin{aligned}\sum c_\beta \mathbf{x}^\beta = &\sum c_\beta (|\mathfrak{p}|x_i)^{1-\mu_i(\beta)} \frac{|\mathfrak{p}|^{n+1} x_i^n \left(1 - \frac{1}{|\mathfrak{p}|}\right)}{|\mathfrak{p}|^{n+1} x^n - 1} |\mathfrak{p}|^{d_i(\beta)-2\beta_i} \mathbf{x}^{\sigma_i \beta} \\ &- \sum c_\beta g^*(1, \epsilon^{-\mu_i(\sigma_i\cdot\beta)}, \mathfrak{p})(|\mathfrak{p}|x_i)^{1-n} \frac{|\mathfrak{p}|^{n+1} x_i^n (1 - |\mathfrak{p}|^n x_i^n)}{|\mathfrak{p}|^{n+1} x^n - 1} p^{d_i(\beta)-2\beta_i} \mathbf{x}^{\sigma_i \beta}.\end{aligned} \tag{3.24}$$

Now, by equating coefficients of $x^\beta$, we arrive at a system of linear equations involving the coefficients $c_\beta$. To actually carry this out, we will rewrite equation (3.24) exhibiting the actions defined in equation (3.10) and equation (3.11) on $\Lambda$:

$$\begin{aligned}\sum c_\beta |\mathfrak{p}|^{n+1}\mathbf{x}^{\beta+n\alpha_i} - c_\beta \mathbf{x}^\beta = \\ \sum c_\beta \left(|\mathfrak{p}|^{n\ell_i(\beta)+n+1}\mathbf{x}^{\sigma_i\star\beta+n\alpha_i} - |\mathfrak{p}|^{n\ell_i(\beta)+n}\mathbf{x}^{\sigma_i\star\beta+n\alpha_i}\right) \\ -\sum c_\beta \left(g^*(1,\epsilon^{-\mu_i(\sigma_i\cdot\beta)},\mathfrak{p})|\mathfrak{p}|^{m_i(\beta)+1}\mathbf{x}^{\sigma_i\cdot\beta} - g^*(1,\epsilon^{-\mu_i(\sigma_i\cdot\beta)},\mathfrak{p})|\mathfrak{p}|^{m_i(\beta)+n+1}\mathbf{x}^{\sigma_i\cdot\beta+n\alpha_i}\right).\end{aligned} \tag{3.25}$$

Finally, equation (3.25) exhibits six terms involving coefficients $c_\beta$ and we equate coefficients of $x^\beta$ by observing that the actions $\sigma_i\cdot$ and $\sigma_i\star$ are involutions. Thus, a linear system for the coefficients $c_\beta$ is given by

$$\begin{aligned}|\mathfrak{p}|^{n+1}c_{\beta-n\alpha_i} - c_\beta - c_{\sigma_i\star\beta+n\alpha_i}|\mathfrak{p}|^{-n\ell_i(\beta)-n+1} + c_{\sigma_i\star\beta+n\alpha_i}|\mathfrak{p}|^{-n\ell_i(\beta)-n} \\ + c_{\sigma_i\cdot\beta}g^*(1,\epsilon^{-\mu_i(\beta)},\mathfrak{p})|\mathfrak{p}|^{-m_i(\beta)+1} - c_{\sigma_i\cdot\beta+n\alpha_i}g^*(1,\epsilon^{-\mu_i(\beta)},\mathfrak{p})|\mathfrak{p}|^{-m_i(\beta)-n+1} = 0,\end{aligned} \tag{3.26}$$

where $\beta$ is any element of $\Lambda$.

This equation generalizes the equation (3.3) in [CFG08]. To compare the two equations specialize equation (3.26) by setting $n = 2$, replace $|\mathfrak{p}|$ with $q$, and replace $x_i$ by $\frac{x_i}{\sqrt{q}}$.

**Theorem 3.4.1.** *The support of the numerator $N(\mathbf{x})$ is contained in the convex hull of the vertices $\{\rho - w\rho | w \in W\}$.*

*Proof.* The proof is computationally identical to that of Theorem 3.2 in [CFG08], although our assumptions here differ somewhat from [CFG08]. By assumption, we are seeking $N(\mathbf{x})$ to have no polar terms since we are only interested in invariant functions with the given denominator $D(\mathbf{x})$. Thus, we know that $c_\beta = 0$ if $\beta \not\succ 0$.

The proof in [CFG08] proceeds along these lines. Inductively, over the length $\ell(w)$ of elements in the Weyl group, we can impose bounds on the support of $N(\mathbf{x})$. The linear equation (3.26) does not eliminate solutions with larger support, but it does show that such solutions would imply a infinite sequence of non-zero terms emanating from the origin. Such a series would be a geometric series and imply that $f(\mathbf{x})$ has additional poles not accounted for in the denominator $D(\mathbf{x})$. □

Chinta and Gunnells in [CGa] prove that there exists a rational function that is invariant under the functional equation described in equation (3.16). This implies that the linear system in equation (3.26) is consistent and a solution exists. We show in the following corollaries that such a solution is unique and derive precisely what it must be:

**Corollary 3.4.2.** *For the stable case, the support of the numerator, $N(x)$, is precisely the vertices $\{\rho - w\rho | w \in W\}$.*

*Proof.* We have shown in the proof of Theorem 3.4.1 that the terms associated to vertices outside the convex hull of $\{\rho - w\rho | w \in W\}$ are $0$.

Let $\beta \in \Lambda$ be such that $\beta$ is not outside the convex hull of $\{\rho - w\rho | w \in W\}$ and $\beta \notin \{\rho - w\rho | w \in W\}$. Then, there exists $w, v \in W$ such that $w \cdot \beta = v \cdot \beta$. Without loss of generality assume that $\beta = \sigma_i \beta$. Then equation (3.26) applies and

$$-c_\beta - c_\beta |\mathfrak{p}|^{-1} = 0,$$

which implies that $c_\beta = 0$. Since all $c_\gamma$ with $\gamma$ in the same orbit of $\beta$ are constant multiples of $c_\beta$, the proof is complete. □

**Corollary 3.4.3.** *For any $n$ and $\beta \in \{w \cdot 0 | w \in W\}$ such that $\sigma_i \cdot \beta \succeq \beta$, we have*

$$c_{\sigma_i \cdot \beta} = c_\beta g(1, \epsilon^{-\mu_i(\beta)}, p) p^{-m_i(\beta)+1}. \tag{3.27}$$

*Proof.* Due to the previous corollary, we only need to consider $\beta$ such that $\beta \in \{\rho - w\rho | w \in W\}$ and $\sigma_i \cdot \beta \succ \beta$. Substituting such a $\beta$ into equation (3.26) gives the desired result immediately.

□

We can expand this recursively and find an excellent computational method to compute the stable coefficients. To compute all coefficients in the support of $N(\mathbf{x})$ it is sufficient to fix the normalization by setting $c_0 = 1$ and enumerate all elements $w \in W$ in order of non-decreasing length. Then, every coefficient can be computed by the multiplication of a single new Gauss sum and power of $|\mathfrak{p}|$ with an already known coefficient. This is exactly the rational function given in equation 3.3.

**Theorem 3.4.4.** *The rational function in equation 3.3 matches the $\mathfrak{p}$-part of the Weyl group multiple Dirichlet series described in [BBC$^+$06].*

*Proof.* Near the end of Section 2 of [BBC$^+$06], the coefficients are defined as

$$H(\mathfrak{p}^{k_1}, \ldots, \mathfrak{p}^{k_r}) = \prod_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Phi^-}} g(\mathfrak{p}^{d(\alpha)-1}, \mathfrak{p}^{d(\alpha)}). \tag{3.28}$$

It is clear that this definition has the same number of Gauss sums as the definition in equation (3.27). They are, in fact, the same Gauss sums. If $\beta = w \cdot 0$, then

$$m_i(\beta) = d\left((\rho - w\rho) - (\rho - \sigma_i w\rho)\right) = d(\sigma_i w\rho - w\rho) = w_j - w_i.$$

Thus, the $m_i(\beta)$ measures the difference of the inverted values in the standard notation for $w\rho$ for the generator $\sigma_i$. This is exactly $d(\alpha)$ for the root $\alpha \in \Phi^+$ such that $\sigma_i w\alpha \in \Phi^-$. □

**Theorem 3.4.5.** *The rational function in equation 3.3 matches the $\mathfrak{p}$-part of the Weyl group multiple Dirichlet series described in [CGa] in the case of the root system $A_r$ stable case.*

*Proof.* Chinta and Gunnells construct a rational function satisfying the functional equation. The previous corollaries explicitly compute the coefficients of the numerator of this rational function and show that such a rational function is unique. Thus, it is forced to be the solution found in [CGa]. □

Together Theorems 3.4.4 and 3.4.5 prove Theorem 3.1.1.

# Chapter 4

# Residues of Weyl Group Multiple Dirichlet Series

*o sum of Gauss!*

*you fugitive,*

*we'll know you in thy properties*

## 4.1 Introduction

We are now ready to fully define the Weyl group multiple Dirichlet series $Z_r^{(n)}$ and state our main result. Define

$$Z_r^{(n)}(s_1, \ldots, s_r) = \Omega(s_1, \ldots, s_r) \sum \frac{H(c_1, \ldots, c_r)}{|c_1|^{s_1}|c_2|^{s_2} \cdots |c_r|^{s_r}} \tag{4.1}$$

where the sum is over all $r$-tuples $(c_1, \ldots, c_r)$ with $c_i$, $1 \leq i \leq r$, monic polynomials in $\mathbb{F}_q[t]$. The product of normalizing zeta factors $\Omega$ is defined

$$\Omega(s_1, \ldots, s_r) = \prod_{\alpha \in \Phi^+} (1 - q^{n|\alpha| - n\sum_{i=1}^r k_i s_i})^{-1}, \tag{4.2}$$

where $\alpha = \sum_{i=1}^r k_i \alpha_i$ (thus $k_i = 0, 1$ for each $1 \leq i \leq r$). The goal of this chapter and this dissertation is to prove the following theorem:

**Theorem 4.1.1.** *Given $Z_r^{(r)}$ defined in equation (4.1) and $Z_{1,FHL}, Z_{2,FHL}$ defined in*

*Chapter 2, we have*

$$\operatorname*{Res}_{x_2 \to q^{-(r+1)/r}} \cdots \operatorname*{Res}_{x_{r-1} \to q^{-(r+1)/r}} Z_r^{(r)}(x_1, x_2, \ldots, x_r) = \mathcal{E}_r \frac{Z_{1,FHL}(q^{1/r}x_1, q^{1/r}x_r)}{\prod_{i=2}^{r-1}(1 - q^{r-i+2}x_1^r)(1 - q^{r-i+2}x_r^r)} \quad (4.3)$$

*and*

$$\operatorname*{Res}_{x_3 \to q^{-(r+1)/r}} \cdots \operatorname*{Res}_{x_r \to q^{-(r+1)/r}} Z_r^{(r)}(x_1, x_2, \ldots, x_r) = \mathcal{E}_r \frac{Z_{2,FHL}(q^{1/2}x_1, q^{(r+1)/r}x_2)}{\prod_{i=2}^{r-1}\left(1 - q^{r-i+1}x_1^r x_2^r\right)\left(1 - q^{r-i+2}x_2^r\right)} \quad (4.4)$$

*where*

$$\mathcal{E}_r = \frac{\sum_{w \in P} T(w)}{r^{r-2}\prod_{i=2}^{r-2}(1 - q^{i+1})}.$$

*Here $P$ is a parabolic subgroup of $W$ generated by the reflections about the hyperplanes of $r-2$ adjacent simple roots. The constant $\mathcal{E}_r$ is precisely a multiresidue (of all $r-2$ variables) of the Weyl group multiple Dirichlet series $Z_{r-2}^{(r)}$.*

The proof of Theorem 4.1.1 involves showing that $Z_r^{(n)}$ as defined in equation (4.1) has a similar rational function form as $H_r^{(n)}$. Thus we state another result in terms of the $\mathfrak{p}$-part which will prove in detail and Theorem 4.1.1 will be a simple corollary to the Theorem 4.1.2.

In Chapter 3 we have established

$$H_r^{(n)}(X_1, \ldots, X_r; \mathfrak{p}) = \frac{\sum_{w \in S_{r+1}} G(w; \mathfrak{p}) X^{\rho - w\rho}}{\prod_{\alpha \in \Phi^+}(1 - |\mathfrak{p}|^{r|\alpha| - 1} X^{\alpha})}. \quad (4.5)$$

We utilize this rational function to prove the following theorem:

**Theorem 4.1.2.** *Given $H_r^{(r)}$ as defined above and $H_{1,FHL}, H_{2,FHL}$ defined in Chapter*

*2, we have*

$$\operatorname*{Res}_{X_2 \to |\mathfrak{p}|^{-1+\frac{1}{r}}} \cdots \operatorname*{Res}_{X_{r-1} \to |\mathfrak{p}|^{-1+\frac{1}{r}}} (H_r^{(r)} X_1, \ldots, X_r; \mathfrak{p}) = \mathcal{C}_r \frac{H_{1,FHL}(|\mathfrak{p}|^{(r-1)/r} X_1, |\mathfrak{p}|^{(r-1)/r} X_r)}{\prod_{i=2}^{r-1} \left(1 - |\mathfrak{p}|^{r+i-2} X_1^r\right) \left(1 - |\mathfrak{p}|^{r+i-2} X_r^r\right)} \quad (4.6)$$

*and*

$$\operatorname*{Res}_{X_3 \to |\mathfrak{p}|^{-1+\frac{1}{r}}} \cdots \operatorname*{Res}_{X_r \to |\mathfrak{p}|^{-1+\frac{1}{r}}} H_r^{(r)}(X_1, \ldots, X_r; \mathfrak{p}) = \mathcal{C}_r \frac{H_{2,FHL}(|\mathfrak{p}|^{1/2} X_2, |\mathfrak{p}|^{(r-1)/r} X_1)}{\prod_{i=2}^{r-1} \left(1 - |\mathfrak{p}|^{r+i-1} X_1^r X_2^r\right) \left(1 - |\mathfrak{p}|^{r+i-2} X_2^r\right)}, \quad (4.7)$$

*where*

$$\mathcal{C}_r = \frac{\sum_{w \in P} G(w; \mathfrak{p})}{r^{r-2} \prod_{i=2}^{r-2} (1 - |\mathfrak{p}|^{i-1})}.$$

*Here $P$ is a parabolic subgroup of $W$ generated by the reflections about the hyperplanes of $r-2$ adjacent simple roots. The constant $\mathcal{C}_r$ is precisely a multiresidue (of all $r-2$ variables) of the $\mathfrak{p}$-part $H_{r-2}^{(r)}$.*

The majority of the work to prove Theorem 4.1.2 lies in analyzing the Gauss sums in the numerator of $H_r^{(r)}(X_1, \ldots, X_r; \mathfrak{p})$. This is done in Section 4.2 where we will derive certain properties of $\sum_{w \in P} G(w; \mathfrak{p})$ where $P$ is a certain subgroup of $W$ which we will describe later. The rest of the proof requires us to account for all the denominator factors of $H_r^{(r)}(X_1, \ldots, X_r; \mathfrak{p})$ and recognize a polynomial factorization and is given in Section 4.3.

A natural question that arises is how the series $Z_{1,FHL}$ and $Z_{2,FHL}$ can have a group of functional equations of order $32$ while the $Z_3$ series associated with $A_3$ has a group of functional equations of order $48$ (the order of the Weyl group is $24$ and there is an additional reflection since the roots $\alpha_1$ and $\alpha_3$ are indistinguishable). We would naturally expect the order of the smaller group to divide the order of the larger but $32 \nmid 48$. This apparent paradox is explained in section 3 of [BB06] where Brubaker and Bump show that the group of order $32$ is really the subgroup of (indeed, isomorphic to) a wreath product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_2$ where the $\mathbb{Z}_2 \times \mathbb{Z}_2$ arises as the group of functional equations satisfied by $Z_{1,FHL}$ by itself (or, by $Z_{2,FHL}$ by itself).

Recall that

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = (\frac{r}{2}, \frac{r}{2} - 1, \dots, -\frac{r}{2} + 1, -\frac{r}{2}).$$

In what follows, we will identify elements $w \in W$ as permutations $w : \Theta \to \Theta$ where

$$\Theta = \left\{ \frac{r}{2}, \frac{r}{2} - 1, \dots, -\frac{r}{2} + 1, -\frac{r}{2} \right\} = \left\{ \frac{r}{2} - i \mid i \in \mathbb{Z} \text{ and } 0 \leq i \leq r \right\}.$$

If $w \in W$ and $i \in \Theta$, then we write $w_i = w(i)$. In the previous chapter, we have represented the action on the root system as a left action. Here, reflecting this new interpretation of elements of $S_{r+1}$, permutations will act on the right, meaning that function composition is left to right. This choice is made to distinguish this from the Weyl group action which acts by permuting the entries of vectors of $r+1$ tuples.

We compute these residues on a linear translation of the $s_1, \dots, s_r$ that we describe now. Define a translated version $H_r^\star$ of the local part $H_r^{(r)}$ by

$$H_r^\star(t_1, \dots, t_r; \mathfrak{p}) = H_r^{(r)}(s_1 - \frac{r-1}{r}, \dots, s_r - \frac{r-1}{r}; \mathfrak{p}).$$

Let $Y_i = |\mathfrak{p}|^{-t_i}$ for $1 \leq i \leq r$. The purpose of $H^\star$ is that using it one can easily see the factorization that will occur in the numerator when we compute the residues. We also define a renormalized Gauss sum

$$\psi_i = \frac{g(\mathfrak{p}^{i-1}, \epsilon, \mathfrak{p}^i)}{|\mathfrak{p}|^{(r-i)/r}}$$

and a product, $\Psi(w; \mathfrak{p})$, over the Gauss sums appearing in the coefficient of $\mathbf{x}^{\rho - w\rho}$ defined by

$$\Psi(w;\mathfrak{p}) = \prod_{\substack{i<j \\ w_i<w_j}} \psi_{w_j - w_i}.$$

It is clear that if $u, v \in \{1, 2, \ldots, r-1\}$ and $u + v = r$, then $\psi_u \psi_v = 1$ and $\psi_r = -1$. With this notation, we have

$$H_r^\star(t_1, \ldots, t_r; \mathfrak{p}) = \frac{\sum_{w \in S_{n+1}} \Psi(w;\mathfrak{p}) Y^{\rho - w\rho}}{\prod_{\alpha \in \Phi^+} (1 - \mathfrak{p}^{|\alpha|-1} Y^\alpha)}. \tag{4.8}$$

## 4.2 Gauss Sum Invariances over Weyl Group Cosets

This section contains two lemmas, which establish identities about inversions of elements of the symmetric group. With our choice of domain $\Theta$, we say there is an *inversion of* $i$ *and* $j$ *for* $w \in W$ if $i > j$ and $w(i) < w(j)$. Thus, there are no inversions in $w$ if $w(\frac{r}{2}) > w(\frac{r}{2} - 1) > \cdots > w(-\frac{r}{2})$. Note that the products $G(w;\mathfrak{p})$ and $\Psi(w;\mathfrak{p})$ have a factor for each inversion of $w \in W$.

We will now motivate the techniques of this section by looking at the numerator of $H_4^\star$, the translated $\mathfrak{p}$-part of the $Z_4$ series. Let $P = \langle \sigma_2, \sigma_3 \rangle$ be the parabolic subgroup of $W$ that fixes the elements $\frac{r}{2}$ and $-\frac{r}{2}$. In terms of the root system, this subgroup is generated by the reflections of the central $r-2$ simple roots. There are 20 cosets of $P$, $P\pi_{ij}$, which can be indexed by $i, j \in \Theta$ and $i \neq j$. The function $\pi_{ij}$ exchanges $i$ with $\frac{r}{2}$ and $j$ with $-\frac{r}{2}$. These 20 cosets are arranged in Figure 4.1 in a way that suggests our strategy.

We extend the definition of $\Psi$ for subset $U \subset W$ by

$$\Psi(U;\mathfrak{p}) = \sum_{w \in U} \Psi(w;\mathfrak{p}). \tag{4.9}$$

Typically $U$ will be a coset of a parabolic subgroup such as $P$. With this definition, we

| | | | | |
|---|---|---|---|---|
| $(2,\star,\star,\star,-2)$ | $(2,\star,\star,\star,-1)$ | $(2,\star,\star,\star,0)$ | $(2,\star,\star,\star,1)$ | |
| $(1,\star,\star,\star,-2)$ | $(1,\star,\star,\star,-1)$ | $(1,\star,\star,\star,0)$ | | $(1,\star,\star,\star,2)$ |
| $(0,\star,\star,\star,-2)$ | $(0,\star,\star,\star,-1)$ | | $(0,\star,\star,\star,1)$ | $(0,\star,\star,\star,2)$ |
| $(-1,\star,\star,\star,-2)$ | | $(-1,\star,\star,\star,0)$ | $(-1,\star,\star,\star,1)$ | $(-1,\star,\star,\star,2)$ |
| | $(-2,\star,\star,\star,-1)$ | $(-2,\star,\star,\star,0)$ | $(-2,\star,\star,\star,1)$ | $(-2,\star,\star,\star,2)$ |

Figure 4.1: The cosets of the parabolic subgroup $P$ for the series associated with $A_4$

rewrite the numerator of equation (4.8)

$$\sum_{w\in S_{n+1}} \Psi(w;\mathfrak{p})Y^{\rho-w\rho} = \sum_{\substack{i,j\in\Theta\\ i\neq j}} \Psi(P\pi_{ij};\mathfrak{p})Y^{\rho-w\rho}. \tag{4.10}$$

Evaluating the residue of Theorem 4.1.2 will involve specializing $Y_2 = Y_3 = \ldots = Y_{r-1} = 1$ and, thus, we are interested in the coefficients $\Psi(P\pi_{ij};\mathfrak{p})$ in

$$\sum_{\substack{i,j\in\Theta\\ i\neq j}} \sum_{w\in P\pi_{ij}} \Psi(w;\mathfrak{p})Y^{\rho-w\rho}\big|_{Y_2=Y_3=\ldots=Y_{r-1}=1} = \sum_{\substack{i,j\in\Theta\\ i\neq j}} \Psi(P\pi_{ij};\mathfrak{p})Y_1^{r/2-i}Y_r^{j+r/2}. \tag{4.11}$$

Define $\gamma \in S_{n+1}$ by

$$\gamma(i) = \begin{cases} i & \text{if } |i| \neq \frac{r}{2} \\ -i & \text{if } |i| = \frac{r}{2}. \end{cases} \tag{4.12}$$

**Lemma 4.2.1.** *If $w \in S_{r+1}$, then $\Psi(w;\mathfrak{p}) = -\Psi(w\gamma;\mathfrak{p})$.*

*Proof.* Without loss of generality, we can assume that $w^{-1}(\frac{r}{2}) < w^{-1}(-\frac{r}{2})$ since $\gamma = \gamma^{-1}$. If $w^{-1}(-\frac{r}{2}) - w^{-1}(\frac{r}{2}) = 1$ the desired result is clear since the only Gauss sum introduced by $\gamma$ is $\psi_r = -1$.

Otherwise let $i$ be such that $w^{-1}(\frac{r}{2}) < i < w^{-1}(-\frac{r}{2})$. Each choice of $i$ is associated with a pair of Gauss sums in the product $\Psi(w\gamma;\mathfrak{p})$ that are not in the product $\Psi(w;\mathfrak{p})$. The product of these two Gauss sums is

$$\psi_{r/2-w_i}\psi_{w_i-(-r/2)} = 1,$$

since $r/2 - w_i + w_i + r/2 = r$. In this case $\Psi(w\gamma; \mathfrak{p})$ also includes $\psi_r$ that does not appear in $\Psi(w; \mathfrak{p})$. Every extra Gauss sum in $\Psi(w\gamma; \mathfrak{p})$ that does not appear in $\Psi(w; \mathfrak{p})$ is accounted for in one of these ways.

Thus $\Psi(w; \mathfrak{p}) = \psi_r \Psi(w\gamma; \mathfrak{p})$, which establishes the lemma. □

There are several immediate conclusions about the coefficients $\Psi(P\pi_{ij}; \mathfrak{p})$ in equation (4.11) that follow from this lemma. First, if $\{i, j\} \cap \{\frac{r}{2}, -\frac{r}{2}\} = \varnothing$, then $\Psi(P\pi_{ij}; \mathfrak{p}) = 0$ since $\{w, w\gamma\} \subset P\pi_{ij}$. This shows that the coefficients corresponding to the six cosets in the central square of Figure 4.1 are $0$. Next, if $|j| < \frac{r}{2}$ then $P\pi_{\frac{r}{2},j}\gamma = P\pi_{-\frac{r}{2},j}$ and $\Psi(P\pi_{\frac{r}{2},j}; \mathfrak{p}) = -\Psi(P\pi_{-\frac{r}{2},j}; \mathfrak{p})$. This shows that the coefficients represented in the left column are negatives of the coefficients represented in the right column. Similarly, if $|i| < \frac{r}{2}$, then $\Psi(P\pi_{i,-\frac{r}{2}}; \mathfrak{p}) = -\Psi(P\pi_{i,\frac{r}{2}}; \mathfrak{p})$. This shows that the coefficients represented in the top row are negations of the coefficients directly below them at the bottom. Lastly, $\Psi(\pi_{\frac{r}{2},-\frac{r}{2}}; \mathfrak{p}) = -\Psi(\pi_{-\frac{r}{2},\frac{r}{2}}; \mathfrak{p})$ showing that the top left coefficient is the negative of the bottom right coefficient.

To finish the proof, we need to show that the coefficients represented in the left column and top row are all equal. In other words, we must show that $\Psi(P\pi_{i,-\frac{r}{2}}; \mathfrak{p})$ and $\Psi(P\pi_{\frac{r}{2},j}; \mathfrak{p})$ are independent of $i$ and $j$. That is the goal of the next lemma.

Define permutations $\tau$ and $\eta$ by

$$\tau(i) = \begin{cases} \frac{r}{2} & \text{if } i = -\frac{r}{2} + 1 \\ -\frac{r}{2} & \text{if } i = -\frac{r}{2} \\ i - 1 & \text{otherwise,} \end{cases} \quad \text{and } \eta(i) = \begin{cases} \frac{r}{2} & \text{if } i = \frac{r}{2} \\ -\frac{r}{2} & \text{if } i = \frac{r}{2} - 1 \\ i + 1 & \text{otherwise.} \end{cases} \tag{4.13}$$

**Lemma 4.2.2.** *If $w \in S_{r+1}$ and $-\frac{r}{2}$ is fixed by $w$, then $G(w\tau; \mathfrak{p}) = G(w; \mathfrak{p})$.*

*Analogously, if $w \in S_{r+1}$ and $\frac{r}{2}$ is fixed by $w$, then $G(w\eta; \mathfrak{p}) = G(w; \mathfrak{p})$.*

*Proof.* The two statements are equivalent by the symmetry of the definitions. We provide details for the first.

The inversions represented in the product $G(w;\mathfrak{p})$ that do not involve $-\frac{r}{2}+1$ are preserved in $G(w\tau;\mathfrak{p})$. For each $i$ with $0 \leq i < \lfloor\frac{r-1}{2}\rfloor$, consider the pair $\frac{r}{2}-i$ and $-\frac{r}{2}+2+i$. We will show that inversions are introduced in canceling pairs in transforming $G(w;\mathfrak{p})$ to $G(w\tau;\mathfrak{p})$, that they are annihilated in pairs, or that they complement each other on either side of the equal sign.

Consider the following four cases:

*Case 1:* $w^{-1}(\frac{r}{2}-i) < w^{-1}(-\frac{r}{2}+1)$ and $w^{-1}(-\frac{r}{2}+2+i) < w^{-1}(-\frac{r}{2}+1)$

The product $G(w\tau;\mathfrak{p})$ includes the pair

$$\psi_{\frac{r}{2}-(\frac{r}{2}-i-1)}\psi_{\frac{r}{2}-(-\frac{r}{2}+2+i-1)} = \psi_{i+1}\psi_{r-i-1} = 1$$

that does not appear in $G(w;\mathfrak{p})$.

*Case 2:* $w^{-1}(\frac{r}{2}-i) > w^{-1}(-\frac{r}{2}+1)$ and $w^{-1}(-\frac{r}{2}+2+i) > w^{-1}(-\frac{r}{2}+1)$

The product $G(w;\mathfrak{p})$ includes the pair

$$\psi_{\frac{r}{2}-i-(-\frac{r}{2}+1)}\psi_{-\frac{r}{2}+2+i-(-\frac{r}{2}+1)} = \psi_{r-i-1}\psi_{i+1} = 1$$

that does not appear in $G(w\tau;\mathfrak{p})$.

*Case 3:* $w^{-1}(\frac{r}{2}-i) < w^{-1}(-\frac{r}{2}+1) < w^{-1}(-\frac{r}{2}+2+i)$

The product $G(w;\mathfrak{p})$ includes the factor $\psi_{\frac{r}{2}-(\frac{r}{2}-i-1)} = \psi_{i+1}$ and $G(w\tau;\mathfrak{p})$ includes the same factor $\psi_{-\frac{r}{2}+2+i-(-\frac{r}{2}+1)} = \psi_{i+1}$.

*Case 4:* $w^{-1}(\frac{r}{2}-i) > w^{-1}(-\frac{r}{2}+1) > w^{-1}(-\frac{r}{2}+2+i)$

The product $G(w;\mathfrak{p})$ includes the factor $\psi_{\frac{r}{2}-i-(-\frac{r}{2}+1)} = \psi_{r-i-1}$ and $G(w\tau;\mathfrak{p})$ includes the same factor $\psi_{\frac{r}{2}-(-\frac{r}{2}+2+i-1)} = \psi_{r-i-1}$.

If $r$ is even, there is also the possibility that either $w$ or $w\tau$ contain an inversion introducing a factor $\psi_{\frac{r}{2}}$ that would not appear in the other. However, $\psi_{\frac{r}{2}} = 1$ so this does not affect the equality. In the context of the pairing argument above, this can be interpreted as a degenerate pair where $\frac{r}{2} - i = -\frac{r}{2} + 2 + i$. $\square$

We now turn our attention to the multiresidue of $H^{\star}$ that yields (a translated version of) $H_{2,FHL}$. The Gauss sum identities required are quite similar to the identities required for the multiresidue yielding $H_{1,FHL}$, which we have already outlined. Figure 4.2 shows the cosets of the parabolic subgroup $R = \langle \sigma_3, \ldots, \sigma_r \rangle$. Note that the entries in Figure 4.2 are arranged so that the $i^{\text{th}}$ column corresponds to terms exactly divisible by $x_1^i$ and the $j^{\text{th}}$ row corresponds to terms exactly divisible by $x_2^j$.

$$\begin{array}{c|cccc|c}
(2,1,\star,\star,\star) & (1,2,\star,\star,\star) & & & & \\
(2,0,\star,\star,\star) & & (0,2,\star,\star,\star) & & & \\
(2,-1,\star,\star,\star) & (1,0,\star,\star,\star) & (0,1,\star,\star,\star) & (-1,2,\star,\star,\star) & & \\
(2,-2,\star,\star,\star) & (1,-1,\star,\star,\star) & & (-1,1,\star,\star,\star) & & (-2,2,\star,\star,\star) \\
 & (1,-2,\star,\star,\star) & (0,-1,\star,\star,\star) & (-1,0,\star,\star,\star) & & (-2,1,\star,\star,\star) \\
 & & (0,-2,\star,\star,\star) & & & (-2,0,\star,\star,\star) \\
 & & & (-1,-2,\star,\star,\star) & & (-2,-1,\star,\star,\star)
\end{array}$$

Figure 4.2: The cosets of the parabolic subgroup $R$ for the series associated with $A_4$

In terms of Figure 4.2, we use Lemma 4.2.2 to show that the coefficients of $x_2^j$ (in the first column) are all equal. The coefficients represented on the super-diagonal, namely the coefficients of $x_1^{i+1} x_2^i$, are

$$\Psi(P\eta^k \delta_{\frac{r}{2}-k,\frac{r}{2}}; \mathfrak{p}) = \psi_k \Psi(P\eta^k; \mathfrak{p}) = \psi_k \Psi(P; \mathfrak{p}). \tag{4.14}$$

Here $\delta_{i,j}$ denotes the permutation that fixes all elements other than $i, j$ and swaps those two. Taking equation (4.14) with $k = r = 4$ and Lemma 4.2.2, we deduce that the coefficients of $x_1^4 x_2^j$ are the negative of the coefficients of $x_2^j$. Lemma 4.2.1 shows that

$$\Psi(P\eta_k;\mathfrak{p}) = -\Psi(P\eta_k\gamma;\mathfrak{p}), \tag{4.15}$$

which implies that the coefficient of $x_1^{i+1}x_2^i$ is negative that of $x_1^{i+1}x_2^{i+4}$ for $0 < i < r$. The six coefficients represented by the center columns that are not at the top or bottom are $0$ by Lemma 4.2.1 since they correspond to cosets invariant under the action of $\gamma$. This is sufficient to see a factorization of the numerator prerequisite to the residue of Theorem 4.1.1.

## 4.3 Normalizing Residues

In this section, we prove Theorem 4.1.2 and note that Theorem 4.1.1 follows by the formal similarity for the $\mathfrak{p}$-part and the global series in the rational function field.

*Proof.* (of Theorem 4.1.2) The residues in equation (4.6) are all at simple poles so they are computed by

$$\lim_{s_2\to 1-\frac{1}{r}} \cdots \operatorname*{Res}_{s_{r-1}\to 1-\frac{1}{r}} (1-|\mathfrak{p}|^{(r-1)/r-s_2})\cdots(1-|\mathfrak{p}|^{(r-1)/r-s_{r-1}})H_r^{(r)}(s_1,\ldots,s_r;\mathfrak{p}).$$

Both residues in the theorem are proven in a very similar way. We will build on the strategy laid out in Section 4.2 to prove the first. Then, we will point out the novel features about the second and let the details for the reader.

The main part of the proof is to analyze the numerator of $H_r^\star$. As before, let $\pi_{ij}$ be the permutation that exchanges $i$ with $\frac{r}{2}$ and $j$ with $-\frac{r}{2}$. Let

$$\begin{aligned} H^*_{r,num}(Y_1,\ldots,Y_r) &= \sum_{w\in W} \Psi(w;\mathfrak{p})Y^{\rho-w\rho} \\ &= \sum_{\substack{i,j\in\Theta\\ i\neq j}} \sum_{w\in P} \Psi(w\pi_{i,j};\mathfrak{p})Y^{\rho-w\pi_{i,j}\rho} \end{aligned} \tag{4.16}$$

be the numerator of $H_r^\star$. In the residue, we need to evaluate

$$H^*_{r,num}(Y_1, 1, 1, 1, \ldots, 1, Y_r) = \sum_{\substack{i,j \in \Theta \\ i \neq j}} \Psi(P\pi_{i,j}; \mathfrak{p}) Y^{\rho - \pi_{i,j}\rho}. \tag{4.17}$$

As described in Section 4.2, we have

$$H^*_{r,num}(Y_1, 1, 1, 1, \ldots, 1, Y_r) = \Psi(P; \mathfrak{p}) \sum_{\substack{i,j \in \Theta \\ i \neq j}} \epsilon(i,j) Y^{\rho - \pi_{i,j}\rho} \tag{4.18}$$

where

$$\epsilon(i,j) = \begin{cases} 1 & i = \frac{r}{2} \text{ or } j = -\frac{r}{2} \\ -1 & i = -\frac{r}{2} \text{ or } j = \frac{r}{2} \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to recognize the factorization

$$H^*_{r,num}(Y_1, 1, 1, 1, \ldots, 1, Y_r) = \Psi(P; \mathfrak{p})(1 - Y_1 Y_r) \left( \sum_{k=0}^{r-1} Y_1^k \right) \left( \sum_{k=0}^{r-1} Y_r^k \right). \tag{4.19}$$

Let $\alpha \in \Phi^+$. The factors in the denominator are accounted for in the following ways:

- If $\alpha_1 \not\prec \alpha$ and $\alpha_r \not\prec \alpha$, then $1 - |\mathfrak{p}|^{r|\alpha|-1} X^\alpha$ contributes to the constant $\mathcal{C}_r$ in the residue.
- If $\alpha_1 \prec \alpha$ and $\alpha_r \prec \alpha$, then $1 - |\mathfrak{p}|^{r|\alpha|-1} X^\alpha$ contributes to the denominator of $H_{1,FHL}$.
- If $\alpha_1 \prec \alpha$ and $\alpha_r \not\prec \alpha$, then $1 - |\mathfrak{p}|^{r|\alpha|-1} X^\alpha$ factors and is partially canceled by a factor from the numerator.
- If $\alpha_1 \not\prec \alpha$ and $\alpha_r \prec \alpha$, then $1 - |\mathfrak{p}|^{r|\alpha|-1} X^\alpha$ factors and is partially canceled by a factor from the numerator.

The proof of the second residue has two distinctive features. Let $\kappa_{i,j}$ be the permutation that exchanges $i$ with $\frac{r}{2}$ and $j$ with $\frac{r}{2} - 1$. If $j = \frac{r}{2}$, then $\Psi(P\kappa_{i,j}; \mathfrak{p}) =$

$\psi_{j-i}\Psi(P\kappa_{j,i};\mathfrak{p})$ and this introduces non-trivial Gauss sums in the residue. These can be seen to correspond with the Gauss sums that appear in $H_{2,FHL}$. One can compute

$$H^{\star}_{r,num}(Y_1,Y_2,1,1,\ldots) = \Psi(P;\mathfrak{p})\left(\sum_{k=0}^{r-1} Y_1^k\right)\left(1 - Y_1^r Y_2^{r-1} + \sum_{k=1}^{r-1}\psi_k Y_1^i Y_2^{i-1}(1-Y_2)\right). \quad (4.20)$$

The second distinctive feature is that the translation for $H^{\star}_r$ does not cancel the translation of $H_{2,FHL}$ in equation (4.7) as it does for equation (4.6). Thus, the third factor in equation (4.20) can be recognized as the numerator of $H_{2,FHL}$ in equation (2.10) with the variable changes $Y_2 \to X$ and $Y_1 \to |\mathfrak{p}|^{(r-2)/2r}Y$. □

## 4.4 Global Series $Z_r^{(n)}$

In [CGb] Chinta and Gunnells describe functional equations which the global series $Z_r^{(n)}$ must satisfy. In that paper they specialize to the root system $A_2$, but the generalization to $A_r$ is not difficult. Let $\beta \in \mathbb{Z}^r$, we write $(c_1,\ldots,c_r) \sim \beta$ if for all $1 \le i \le r$ we have deg $c_i \equiv \beta_i$ mod $n$. Define

$$Z_r^{(n)}(s_1,\ldots s_r;\beta) = \Omega(s_1,\ldots,s_r)\sum_{(c_1,\ldots,c_r)\sim\beta}\frac{H(c_1,\ldots,c_r)}{|c_1|^{s_1}|c_2|^{s_2}\cdots|c_r|^{s_r}} \quad (4.21)$$

where the sum only includes monic polynomials. The global functional equation is

$$Z_r^{(n)}(\mathbf{x};\beta) = \mathfrak{P}_{\beta,i}(x_i)Z_r^{(n)}(\sigma_i\mathbf{x};\beta) + \mathfrak{Q}_{\sigma_i\cdot\beta,i}(x_i)Z_r^{(n)}(\sigma_i\mathbf{x};\beta), \quad (4.22)$$

where

$$\begin{aligned}\mathfrak{P}_{\beta,i}(x) &= (qx)^{1-\mu_i(\beta)}\frac{1-q}{1-q^{n+1}x^n}, \text{ and}\\ \mathfrak{Q}_{\beta,i}(x) &= -\tau(\epsilon^{-\mu_i(\beta)})(qx)^{1-n}\frac{1-q^n x^n}{1-q^{n+1}x^n}.\end{aligned} \quad (4.23)$$

Since we have a complete understanding of the poles of $Z_r^{(n)}$, the uniqueness of the $\mathfrak{p}$-part established in Chapter 3 applies to the global series as well. Thus, we have the following theorem:

**Theorem 4.4.1.** *The series $Z_r^{(n)}$ is the rational function in the variables $x_i = q^{-s_i}$, $1 \leq i \leq r$, given by*

$$Z_r^{(n)}(x_1, \ldots, x_r) = \frac{\sum_{w \in S_{r+1}} T(w) x^{\rho - w\rho}}{\prod_{\alpha \in \Phi^+}(1 - q^{r|\alpha|+1} x^{\alpha})}, \tag{4.24}$$

*where*

$$T(w) = \prod_{\substack{i<j \\ w_i < w_j}} \tau^*(\epsilon^{w_j - w_i}) q^{w_j - w_i} \tag{4.25}$$

*and*

$$\tau^*(\epsilon^i) = \begin{cases} \tau(\epsilon^i) & \text{if } i \not\equiv 0 \text{ mod } n \\ q\tau(\epsilon^i) & \text{otherwise.} \end{cases} \tag{4.26}$$

*Thus, the variable transformations (1.3) transform $H_r^{(n)}$ to $Z_r^{(n)}$.*

*Proof.* We observe that the variable transformations (1.3) transform $P_{\beta,i}(x)$ and $Q_{\beta,i}(x)$ from equation (3.15) to $\mathfrak{P}_{\beta,i}(x)$ and $\mathfrak{Q}_{\beta,i}(x)$ from equation (4.23) respectively. The expressions on the right hand side of equation (4.24) and $Z_r^{(n)}$ defined in equation (4.1) have the same poles, satisfy the same functional equation, and both have a constant term of $1$. Analogs of Theorem 3.4.1 and Corollaries 3.4.2 and 3.4.3 imply that such an expression must be unique. The second statement of the theorem is now easily verified. □

Finally, we can easily prove the capstone of this dissertation.

*Proof.* (of Theorem 4.1.1) Recall that the variable transformations (2.11) transform $H_{i,FHL}$ to $Z_{i,FHL}$ for $i = 1, 2$. We have just shown that the variable transformations (1.3) transform $H_r^{(n)}$ to $Z_r^{(n)}$. With Theorem 4.1.2 in hand, the residues of Theorem 4.1.1 are forced to hold up to some linear translation of the parameters $s_i$. It is easy to verify that the linear translation is as claimed. □

# Chapter 5

# Conclusion

*an oriole*

*of origami fold*

*unfurl your colored polygons*

It is hoped that the explicit computations of these series in this dissertation can provide insight for more general fields. However, even within the context of the rational function field, we expect to be able to use the methods of this current work to explore conjectural relationships among multiple Dirichlet series.

It should be possible to utilize the theory of Chapter 3 to be able to readily compute stable and unstable (i.e. when $n$ is small compared to the rank of the root system) local factors for Weyl group multiple Dirichlet series. It is believed that the derivation of the linear system in equation (3.26) should generalize in a straightforward way to more general root systems following the definitions in [CGa]. With that generalization we expect that we can find an efficient way to compute local factors for any of the irreducible root systems.

Let $Z_r^{(n)}$ denote the Weyl group multiple Dirichlet series associated to $A_r$ with $n^{\mathrm{th}}$ order Gauss sums. An immediate goal of such efficient computation would be to investigate residues of $Z_r^{(n)}$ for $n < r$. Computational evidence suggests that $(r-1)$-fold residues of $Z_r^{(r-1)}$ results in a product of Riemann zeta functions. A multiresidue of $Z_r^{(n)}$ with $n < r-1$ also appear to reveal some structure.

It is conjectured in [CGa] that a $3$-fold residue of the cubic Weyl group multiple Dirichlet series associated with $E_6$ will recover a series used by Brubaker in [Bru03] to prove an asymptotic formula for the second moment of a cubic Dirichlet $L$-series. This proof should be attainable in general with current techniques since the Fourier coefficient of the cubic theta function is known. Explicit computations with the local factor is an easy way to test the conjecture.

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# Vita

Born: January 9, 1979
Married, Spouse: Lydia; Two Children

**Education:**

- **PhD:** May 2009 Lehigh University
- **MS:** May 2005 Lehigh University
- **BS:** December 2002 Millersville University (Mathematics Departmental Honors and University Honors).

**Selected Speaking:**

- Factoring Multivariate Polynomials - Graduate Student Colloquium at Lehigh (April 2008)
- Sums of $L$-functions over the Rational Function Field - Stanford University (July 2006); Texas A&M University (March 2008)
- A Collatz Sequence Sieve - Graduate Student Colloquium at Lehigh (September 2006)
- The Functional Equation for the Riemann Zeta Function - Graduate Student Colloquium at Lehigh (October 2005)
- Repeating Decimals & Finite Fields - Graduate Student Colloquium at Lehigh (October 2004)

**Writing:**

- *Minimal Paths on Some Simple Surfaces with Singularities* (with Ron Umble). Pi Mu Epsilon Journal, 12(8), 2008.

**Other Experience - Software Development:**

- 2006-present: Contributed significant patches to the SAGE computer algebra system.
- 2005-2006: Contributed a variety of patches to the SCons opensource build tool project written in Python.
- 1996-2003 and 2006-present (summers): Esh Computer Center. I provided technical guidance and implementation for an accounting system comprised of approximately $500,000$ lines of C++ source.